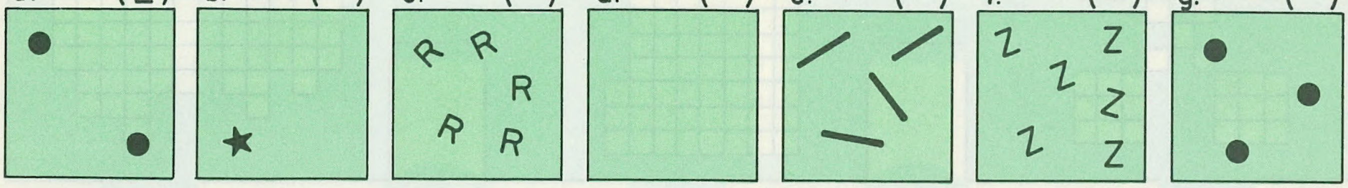
Take a quick glance at one of the sketches below. Now answer the question "How many?" without counting. Glance at the others.

"How many?" In how many cases did you need to count?

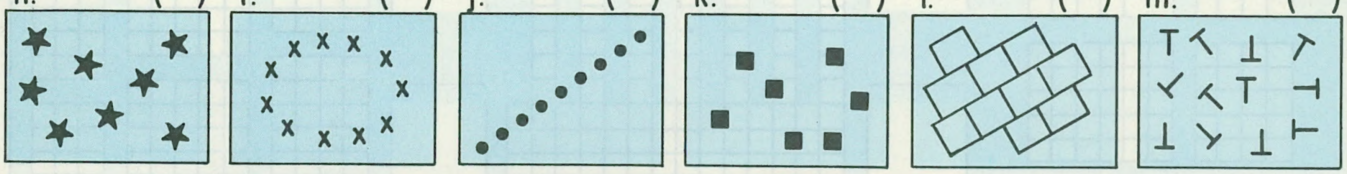
a. (2) b. ( ) c. ( ) d. ( ) e. ( ) f. ( ) g. ( )



Try the same experiment with the next group of sketches. Without counting, answer: "How many?" "An odd or even number?"

"Which sketch shows the largest number of items?" "Which shows the smallest number of items?"

h. ( ) i. ( ) j. ( ) k. ( ) l. ( ) m. ( )

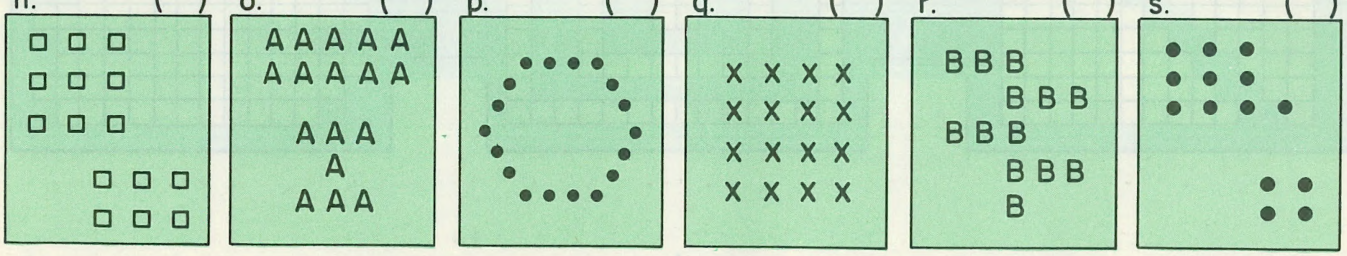


You failed! But don't feel bad. Everyone does unless he has a most unusual kind of vision and memory.

A few persons may have noticed that picture *k* has 7 little squares, but beyond that number of objects you probably needed to count.

Try the same experiment with the sketches below.

n. ( ) o. ( ) p. ( ) q. ( ) r. ( ) s. ( )



Again, a quick glance is not enough. But the special arrangements certainly help! And you probably depended heavily on your experience.

As quickly as possible:

1. Of the 19 sketches above, list those that represent even numbers: d, a, e, ____

____

2. Of the 19 sketches above, list those that represent odd numbers: ____________

____

3. Go back to each sketch, count the number of items in it, and write the result in the parentheses ( ) above each example.

(Those numbers you just wrote will help you with the following.)

4. List all of them that are multiples of 3.

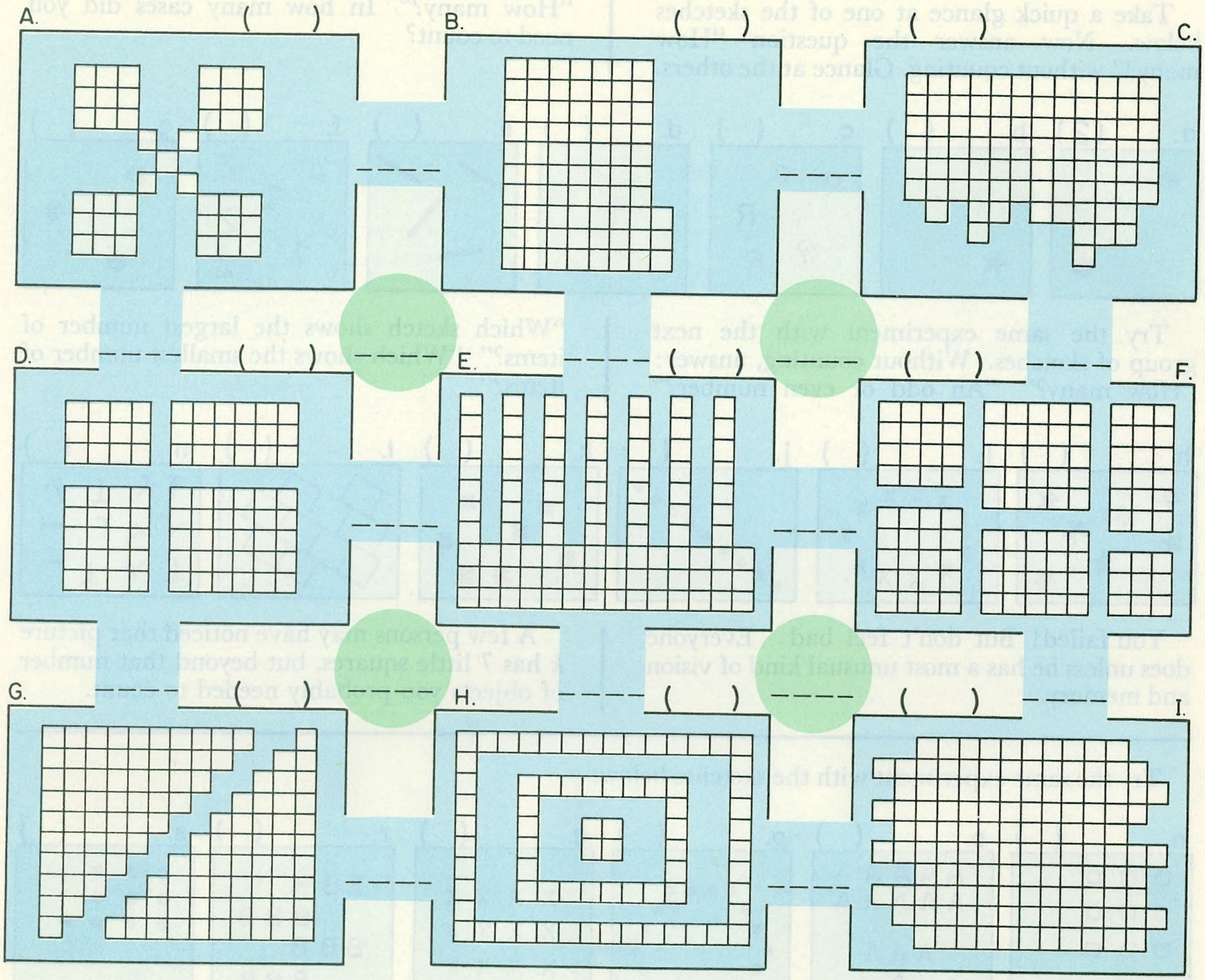
____

5. List all of them that are multiples of 4.

____

Because vision fails us so quickly, we need to be able to count so that we can answer "How many?" And we need to be able to record the results of our counting.

TASK ONE: Please find out how many little squares there are in each sketch, and record your results in the parentheses above each sketch. Look for shortcuts.



TASK TWO: Please find the total number of little squares in each pair of adjacent sketches. Record the results in the blue blocks.

TASK THREE: Please find the total number of little squares in the 4 sketches that touch each of the green tinted discs, and record the number in those discs.

The authors sincerely hope you found many shortcuts so that you didn't have to count all those little squares. We are sure you were glad you kept records of your results. In that way, you didn't have to count the squares in each sketch over and over again.

However, you could have counted the squares, one-by-one, to find correct answers to record.

So, arithmetic also includes:

. . . developing strategies or shortcuts to avoid one-by-one counting.

Arithmetic deals with:
(a) one-by-one counting,
(b) recording results,
(c) shortcuts to avoid one-by-one counting.

And, in the opinion of the authors, that's about all there is to this subject of arithmetic.

Do you agree with us?

If you will look over your work on page 2, you will realize that you seldom depended on one-by-one counting. In each case, you developed your own strategy to save time and to keep from making careless errors that are apt to happen in long counts.

Think About Your Strategies

Patterns

You probably took a glance at each sketch wondering whether it had been drawn in some particular way that would suggest shortcuts. Many patterns include repetitions — and if two or more arrangements are clearly the same size, you need only find out how many there are in one of the arrangements.

Memory

If there is a 3-by-4 chunk, you use your memory to avoid having to count by ones. You remember that there are 12 in such arrangements.

Often, you saw the sketches as several small chunks and used your memory to tell you how many you would find if you counted.

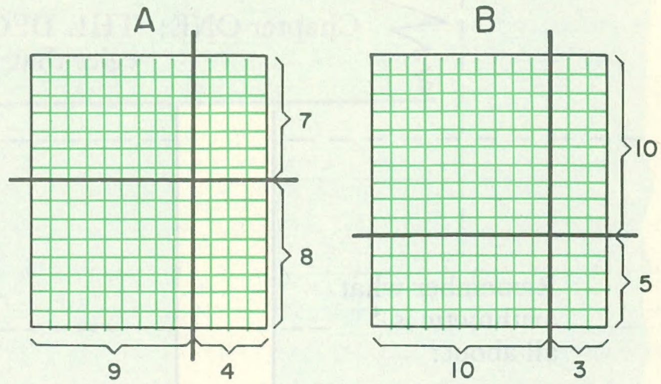
Looking for 10's

You might have looked for tens and multiples of tens, especially for hundreds. And, if the sketches were larger, you would look for multiples of hundreds, thousands, etc., because this leads us to

this ..... rather than	this
300 + 500 = _____	237 + 563 = _____
20 x 90 = _____	24 x 75 = _____
50 x 80 = _____	125 x 32 = _____
10 x 10 x 21 = _____	15 x 35 x 4 = _____

15 rows with 13 in a row?

Few of us remember the product of 13 and 15; so we break it up into *smaller* problems. The sketches below suggest two strategies:



A		B	
4 x 8 =	_____	3 x 5 =	_____
4 x 7 =	_____	3 x 10 =	_____
9 x 8 =	_____	10 x 5 =	_____
9 x 7 =	_____	10 x 10 =	_____
Total	_____	Total	_____

Which strategy do you prefer?

Both are better than finding the number with one-by-one counting!

Looking for 10ths

Looking for tenths, hundredths, and thousandths, etc., is another very fruitful search. It leads to important shortcuts, to easier bookkeeping as we record the results we would have counted if necessary.

The key to developing good strategies to avoid one-by-one counting is a good understanding of the decimal system, the system based on tens and hundreds and on tenths and hundredths of units.

Your authors believe that developing strategies to avoid one-by-one counting is what arithmetic is all about!

100 | 10 | 1 unit | .01 .001 .0001

Chapter ONE: THE DECIMAL SYSTEM — the key to many strategies that help avoid the need for one-by-one counting.

Remember what arithmetic is all about:

COUNTING

RECORDING the results of counting

and

SHORTCUTS to avoid 1-by-1 counting

With only ten digits:

0, 1, 2, 3, 4, 5, 6, 7, 8, and 9

and with the idea of

Place Value

you can write any number you need or wish to write.

It's easy to read large numbers:

$$\underbrace{79}_{\text{Billions}},\underbrace{347}_{\text{Millions}},\underbrace{205}_{\text{Thousands}},870$$

Start reading the number at the left, and use the indicated words as you come to each comma:

"seventy nine *billion,* three hundred forty-seven *million,* two hundred five *thousand,* eight hundred seventy"

. . . unless there are three zeros to the left of a comma. Then, don't say anything when you pass the three zeros and the comma that follows them:

2 , 120 , 000 , 061

Billions Millions Pass over

"two *billion,* one hundred twenty *million,* sixty-one."

Try a few:

48,206	127,138,000
175,175	1,000,001
301,000	100,000,984
800,008	60,001

--- --- ---, --- --- --- . --- --- --- --- --- --- --- --- --- --- ---

"thousand" "and"

tenths, hundredths, thousandths, ten thousandths, hundred thousandths, millionths, ten millionths, hundred millionths, billionths, ten billionths, hundred billionths

The *decimal point* is another bit of shorthand. It is read "and" if the number is greater than 1.

3.7 . . . is read "three *and* seven tenths."

108.18 . . . is read "one hundred eight *and* eighteen hundredths."

8,176.04 . . . is read "eight thousand one hundred seventy-six *and* four hundredths."

When there are no nonzero digits to the left of the decimal point, we do not say "and."

.9 . . . is read "nine tenths."

0.12 . . . is read "twelve hundreths."

.084 . . . is read "eighty-four thousandths."

.1072 . . . is read "one thousand seventy-two ten thousandths."

Caution: Confusion can result if "and" is used for any other reason than to indicate the location of the decimal point. A careless person might read the last example above as

"one thousand and seventy-two ten thousandths" . . . which means

1,000.0072.

I. Read the following expressions:

a.	3.08	*h.*	.1078
b.	18.1	*i.*	.010
c.	4.125	*j.*	2.100
d.	.008	*k.*	4.001
e.	.1009	*l.*	.0807
f.	129.018	*m.*	.19473
g.	27.00012	*n.*	.002009

II. Without using abbreviations or shorthand, write out some of the expressions given at the bottom of the last column.

e. ____________________

____

f. ____________________

____

g. ____________________

____

i. ____________________

l. ____________________

____

n. ____________________

____

III. Could you count by ones to a million? Of course, provided that you had plenty of time and a good reason for doing it.

If a clock makes a "tick" every second, how long does it take for the clock to make a million ticks?

1 in 1 second

60 in 1 minute

____________ in 1 hour

____________ in 1 day

____________ in 1 week

____________ in 1 week and 4 days

____________ in 1 week and 5 days

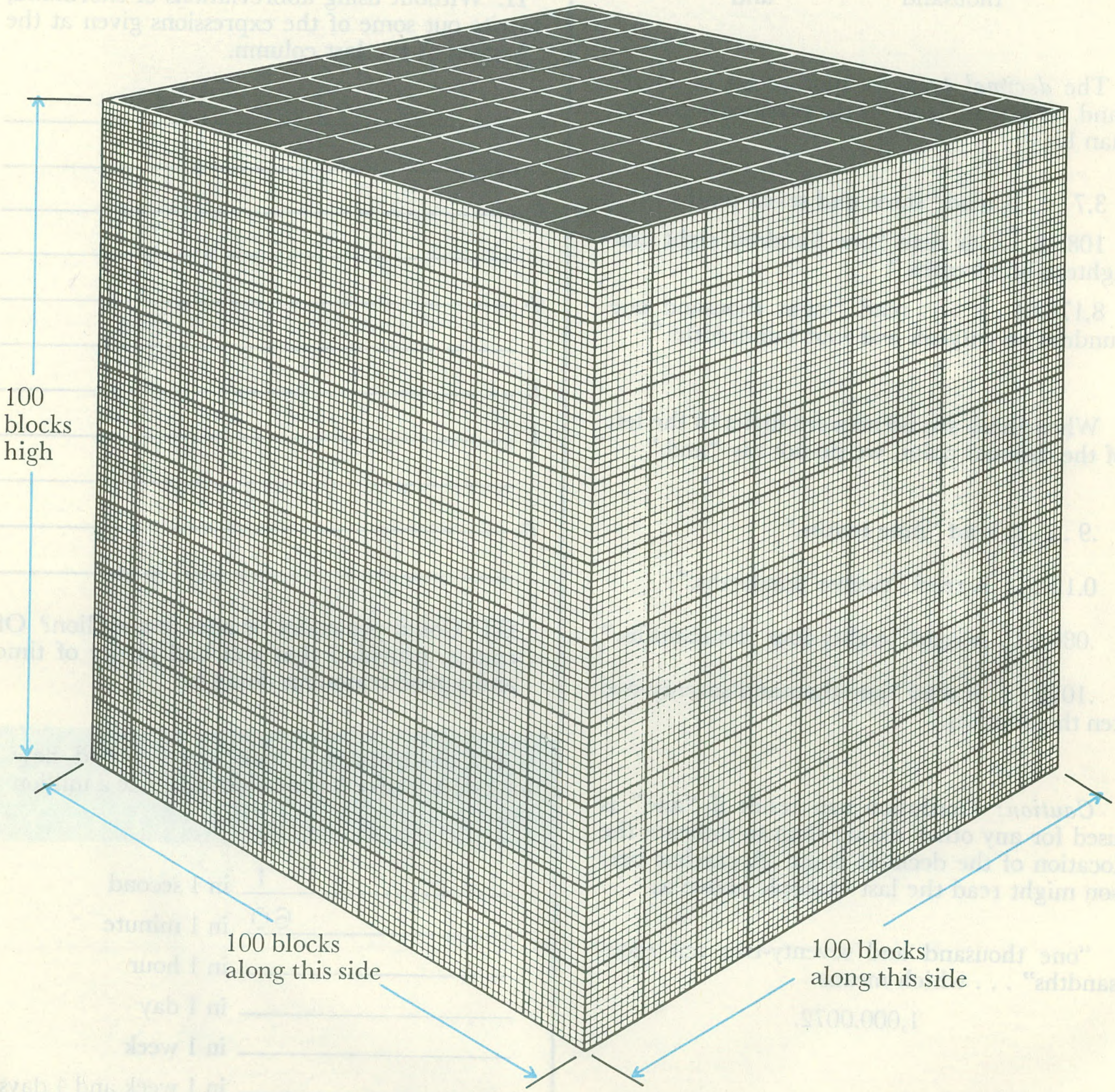
____________ in 365 days

So, big numbers really do arise when we wonder about such questions.

What would a stack of a million blocks look like? We've tried to give you some idea on the next page.

Here's a big stack of blocks.

You would have to count to a million if you wished to count all the blocks in a stack as big as the one illustrated in this drawing.



The bottom layer of tiny blocks would have 100 in each of 100 rows.

How much is 100×100? ______

There would be 100 such layers.

How many in 100 layers?

$100 \times 100 \times 100 =$ ______

Of course, you see only a few of the blocks in this stack — those on the outside.

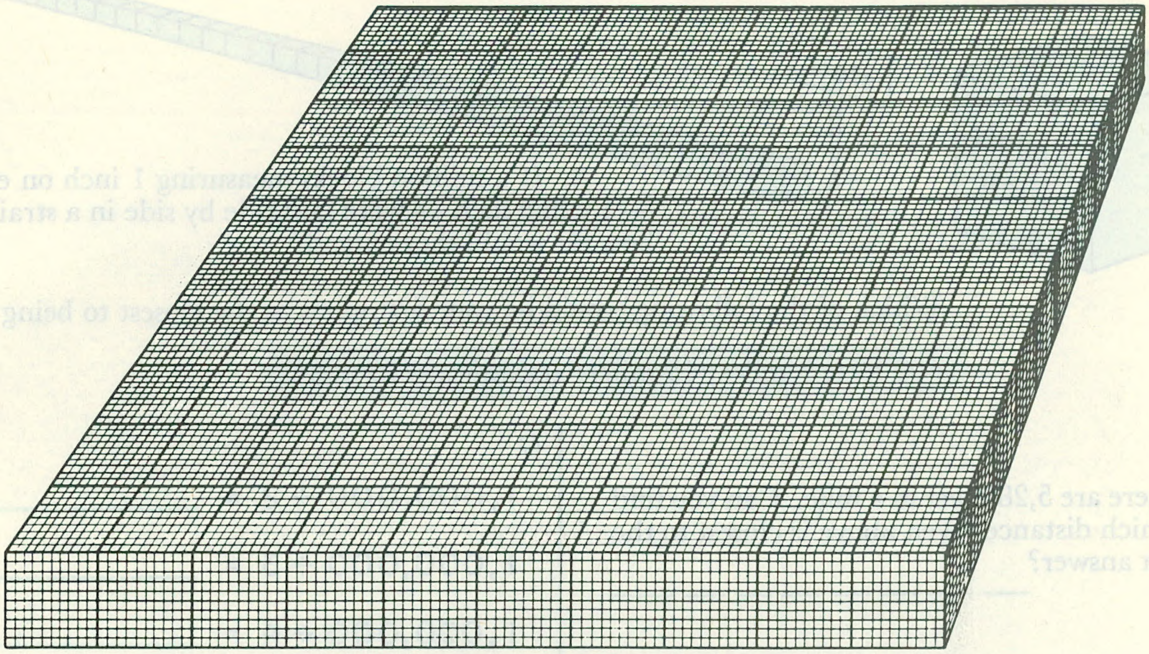
1,000,000 blocks

. . . one million of them!

Do you think that there are a million grains in a bushel of wheat? ______

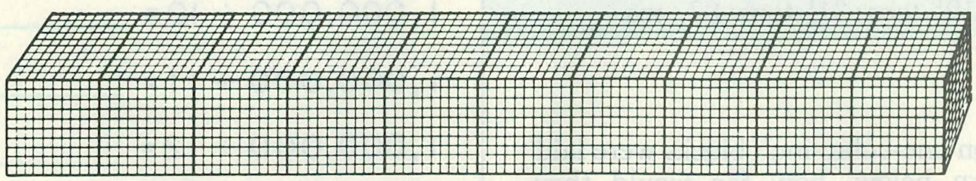
Believe it or not, there are about 5,000,000 grains in one bushel of wheat.

Ten layers of the tiny blocks . . . with 100 × 100 in each layer.



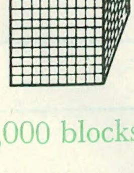
100,000 blocks . . . one hundred thousand of them . . . a tenth of the big stack.

100 × 100 is 10,000. Again, you see only one side of some, two sides of others, and three sides of one. (When you have a little time to spare, you might like to determine how many of these hundred thousand blocks are showing in the picture.)



10,000 blocks . . . ten thousand of them . . . a hundredth of the big stack.

a thousandth of the big stack



1,000 blocks

a ten-thousandth of the big stack

100 blocks

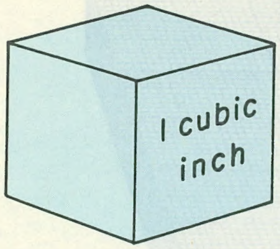
a hundred-thousandth of the big stack

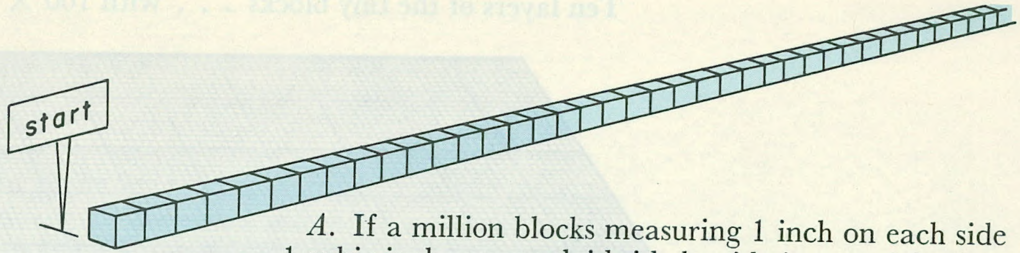
10 blocks

a millionth of the big stack

1 block

Each sketch illustrates a stack that has 10 times as many blocks as the sketch that follows it . . . or, starting with the single block at the right, each sketch illustrates $\frac{1}{10}$ of the next larger stack of blocks.





A. If a million blocks measuring 1 inch on each side — 1 cubic inch — were laid side by side in a straight line, how far would they extend?

Which of the following would be your first guess as the closest to being correct?

500 feet; 5 miles; 15 miles; 100 miles; 770 miles

B. There are 5,280 feet in a mile. Can you find out which distance given above is closest to the correct answer?

____

C. If each one-cubic-inch block of a certain material weighs 1 ounce, how many tons would a million of those blocks weigh? (There are 16 ounces in a pound and there are 2,000 pounds in a ton.)

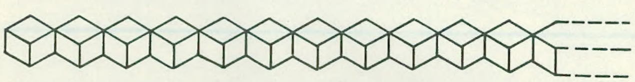
____

Which of the weights below is closest to being the correct weight?

1 ton; $19\frac{1}{2}$ tons; $31\frac{1}{4}$ tons; 87 tons

D. If a million one-cubic-inch blocks were laid in the pattern below, how far would they reach?

____



15 miles; 19 miles; 22 miles; 25 miles; 30 miles

E. How many running feet of 2″ × 4″ blocks can be cut from 1,000,000 cubic inches of wood?

____

F. How many square feet could be covered by 1,000,000 cubes of wood that are 1″ by 1″ by 1″?

____

G.

1,000,000 ÷ 2 = __________

1,000,000 ÷ 3 = __________

1,000,000 ÷ 4 = __________

1,000,000 ÷ 5 = __________

1,000,000 ÷ 6 = __________

1,000,000 ÷ 7 = __________

1,000,000 ÷ 8 = __________

1,000,000 ÷ 9 = __________

1,000,000 ÷ 10 = __________

1,000,000 ÷ 11 = __________

1,000,000 ÷ 12 = __________

1,000,000 ÷ 13 = __________

1,000,000 ÷ 14 = __________

1,000,000 ÷ 15 = __________

1,000,000 ÷ 20 = __________

1,000,000 ÷ 25 = __________

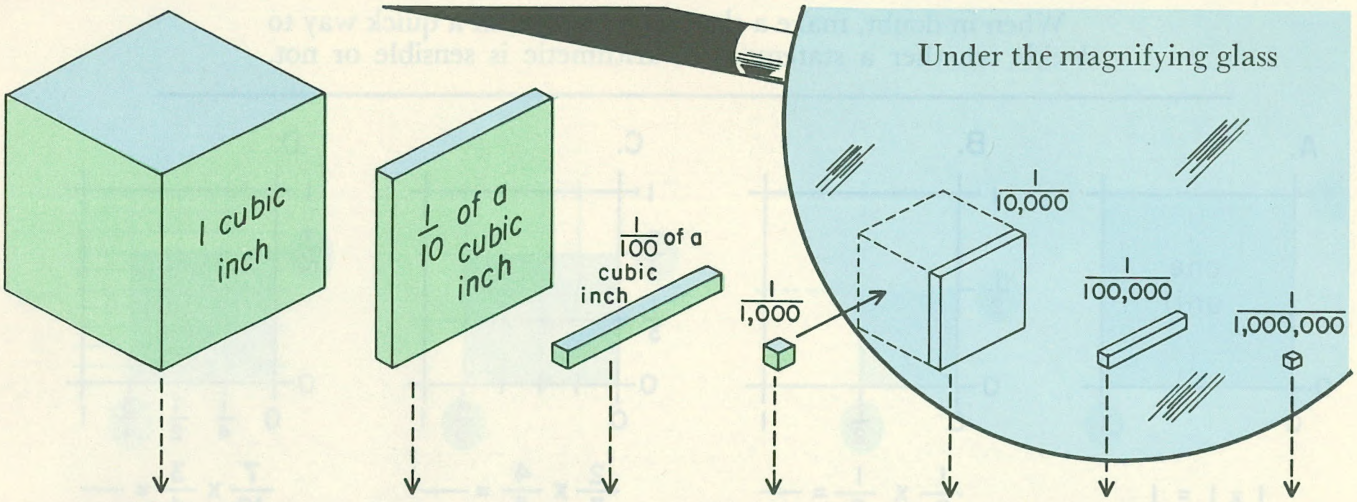
1,000,000 ÷ 75 = __________

1,000,000 ÷ 100 = __________

1,000,000 ÷ 500 = __________

1,000,000 ÷ 750 = __________

1,000,000 ÷ 1,000 = __________



The weight of each piece above in ounces, if a 1-cubic-inch block weighs 1 ounce:

1	.1	.01	____	____	____	____

Does one-millionth of a cubic inch seem very small? Each side would measure .01 of an inch. Do you think you could see it without a magnifying glass? ______

It might look like a grain of sand. But there are things much smaller. If this tiny piece, one-millionth of a cubic inch, were hollow, it could hold many bacteria — about 15 million of them. Molecules are much smaller than bacteria. Atoms are still smaller. Electrons are smaller still.

I. In arithmetic, *of* and *times* indicate the same operation. We also have several ways to write fractions. Complete the following:

a. $\frac{1}{10}$ of $1 = \frac{1}{10}$	.1 of 1 = ____
b. $\frac{1}{10}$ of $\frac{1}{10}$ = ____	.1 of .1 = ____
c. $\frac{1}{10}$ of $\frac{1}{100}$ = ____	.1 x .01 = ____
d. $\frac{1}{100} \times \frac{1}{100}$ = ____	.01 x .01 = ____
e. $\frac{1}{10} \times \frac{1}{1,000}$ = ____	.1 x .001 = ____
f. $\frac{2}{10} \times \frac{2}{10}$ = ____	.2 x .2 = ____
g. $\frac{2}{10} \times \frac{6}{10}$ = ____	.2 x ____ = ____
h. $\frac{5}{100} \times \frac{5}{100}$ = ____	.05 x ____ = ____
i. $\frac{3}{10} \times \frac{15}{1,000}$ = ____	.3 x ____ = ____

II. Some ways to express the number 1:

a. .3 + .7	i. .1 x 10	q. .7 ÷ .7
b. 1.4 − ____	j. ____ ÷ .19	r. ____ + .08
c. ____ + 1.0	k. ____ − .012	s. .25 x ____
d. ____ x 5	l. .07 + ____	t. 10 x ____
e. 8 − ____	m. ____ x .01	u. 2.3 − ____
f. 100 x ____	n. ____ + .19	v. ____ ÷ 93
g. ____ − 1.81	o. ____ x .5	w. ____ + .84
h. ____ x .025	p. .007 + ____	x. .02 x ____

III. Some ways to express .1:

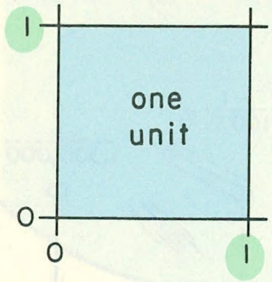
a. .2 ÷ ____	d. ____ + .06	g. ____ − 1.3
b. 1 x ____	e. .2 x ____	h. ____ ÷ 10
c. ____ x .001	f. ____ x 10	i. .011 + ____

IV. Some ways to express .01:

a. .01 + ____	e. ____ x 1	i. .8 − ____
b. .001 + ____	f. 10 x ____	j. 5 x ____
c. ____ − .123	g. .009 + ____	k. ____ x .0025
d. 2 x ____	h. ____ + .0001	l. 100 x ____

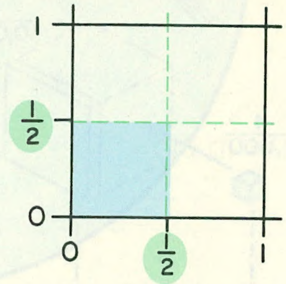
When in doubt, make a sketch. That's often a quick way to decide whether a statement in arithmetic is sensible or not.

A.



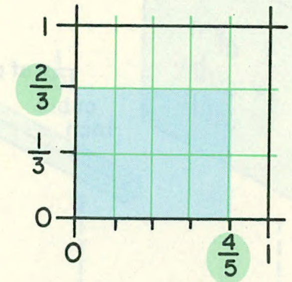
$1 \times 1 = 1$

B.



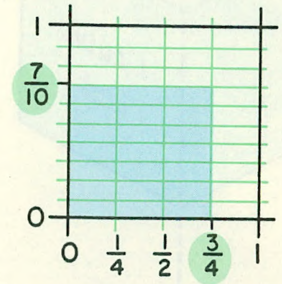
$\frac{1}{2} \times \frac{1}{2} = \_\_$

C.



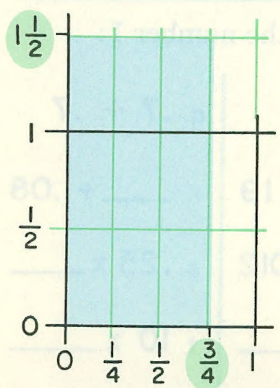
$\frac{2}{3} \times \frac{4}{5} = \_\_$

D.



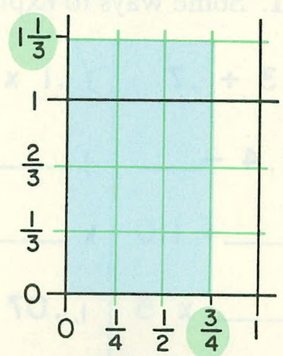
$\frac{7}{10} \times \frac{3}{4} = \_\_$

E.



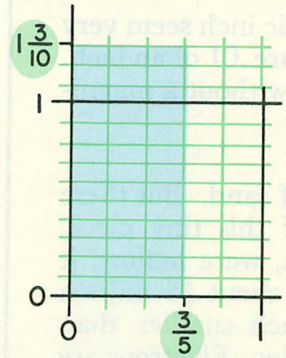
$\frac{3}{2} \times \frac{3}{4} = \_\_ =$

F.



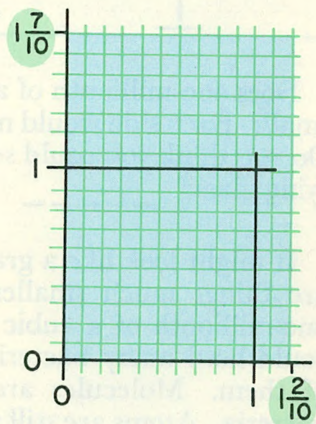
$\_\_ \times \_\_ = \_\_ =$

G.



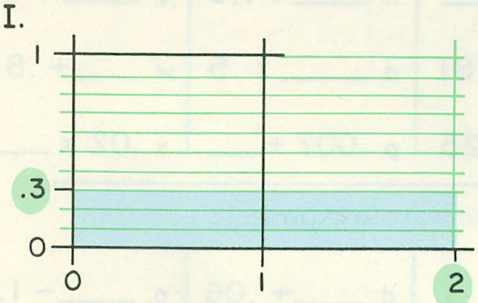
$\_\_ \times \_\_ = \_\_$

H.



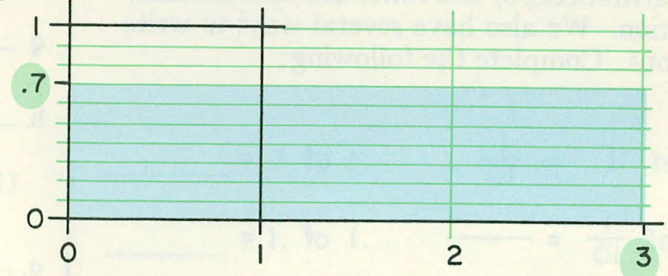
$\_\_ \times \_\_ = \_\_ =$

I.



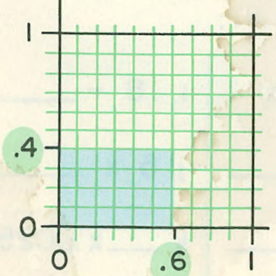
.3 x 2 = 2 x .3 =

J.



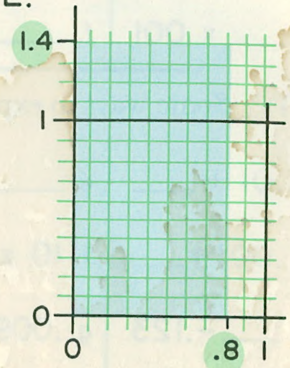
.7 x 3 = 3 x .7 =

K.



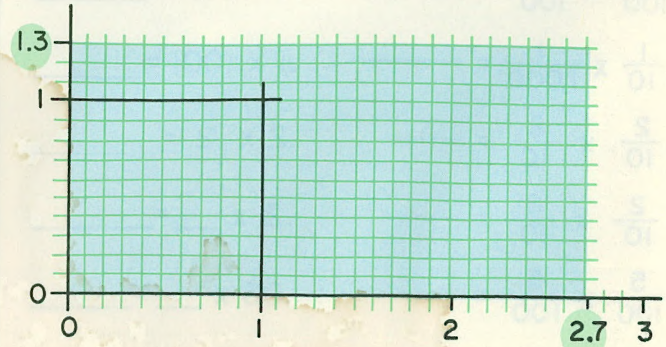
.4 x .6 =

L.



1.4 x .8 =

M.



1.3 x 2.7 =

A. Engineers usually use "decimal equivalents" in noting dimensions that include fractions of inches. They write halves, fourths, and eighths as thousandths. Often they write whole numbers with 3 zeros to the right of the decimal point. Odd 16ths and 32nds of an inch require 4 and 5 places. You can make your own table:

1.000	
.	$\frac{31}{32}$
.	$\frac{15}{16}$
.	$\frac{29}{32}$
.	$\frac{7}{8}$
.84375	$\frac{27}{32}$
.	___
.	$\frac{25}{32}$
.	___
.	$\frac{23}{32}$
.	___
.	$\frac{21}{32}$
.	___
.	$\frac{19}{32}$
.	___
.	$\frac{17}{32}$
.500	$\frac{1}{2}$
.	$\frac{15}{32}$
.	___
.	$\frac{13}{32}$
.	___
.	$\frac{11}{32}$
.	___
.	$\frac{9}{32}$
.250	$\frac{1}{4}$
.	$\frac{7}{32}$
.	___
.	$\frac{5}{32}$
.	$\frac{1}{8}$
.09375	$\frac{3}{32}$
.	$\frac{1}{16}$
.	$\frac{1}{32}$

B. You have probably used the shorthand for "squaring" and, perhaps, "cubing" a number.

$7^2 = 7 \times 7 =$ 49 $\qquad 7^3 = 7 \times 7 \times 7 =$ _____

$4^2 = 4 \times 4 =$ _____ $\qquad 4^3 = 4 \times 4 \times 4 =$ _____

$2^2 = 2 \times 2 =$ _____ $\qquad 2^3 = 2 \times 2 \times 2 =$ _____

$9^2 = 9 \times 9 =$ _____ $\qquad 9^3 = 9 \times 9 \times 9 =$ _____

$10^2 = 10 \times 10 =$ _____ $\qquad 10^3 = 10 \times 10 \times 10 =$ _____

$5^4 = 5 \times 5 \times 5 \times 5 =$ _________

$8^4 = 8 \times 8 \times 8 \times 8 =$ _________

$6^4 = 6 \times 6 \times 6 \times 6 =$ _________

$10^4 = 10 \times 10 \times 10 \times 10 =$ _________

$10^5 = 10 \times 10 \times 10 \times 10 \times 10 =$ ___________

$10^6 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 =$ _________

$10^7 =$ 10,000,000 $\qquad 10^8 =$ ___________

$10^9 =$ ___________ $\qquad 10^{10} =$ ___________

$10^{15} =$ ______________________

Scientists use this shorthand because it helps indicate very large numbers in such a short space. They read 10^{15} as "ten to the 15th power" or simply as "ten to the 15th."

They extend this system, usually referred to as "scientific notation," to express numbers that are not powers of 10. Here is the plan:

$700 = 7 \times 100 = 7 \times 10^2$

$19{,}000 = 19 \times$ 1,000 $= 19 \times$ 10^3

$330{,}000 =$ 33 x ______ = ___x___

$6{,}000 = 6 \times 10^3$ $\qquad 2{,}400 =$ ______

$500{,}000 =$ ______ $\qquad 31{,}000{,}000 =$ ______

$80{,}000 =$ ______ $\qquad 30{,}000{,}000 =$ ______

Often the plan is extended in this way:

$1{,}200 = 12 \times 100 = 1.2 \times 1{,}000 = 1.2 \times 10^3$

$87{,}000 = 87 \times 1{,}000 = 8.7 \times 10{,}000 =$ _______

$12{,}500 = 125 \times 100 = 1.25 \times 10{,}000 = 1.25 \times 10^4$

$3{,}870 = 387 \times 10 = 3.87 \times 1{,}000 =$ ________

$15{,}000 = 1.5 \times$ _____

$7{,}180{,}000 = 7.18 \times$ _____

Astronomers have observed stars that are

2,500,000,000,000,000 miles

from the earth. We save a lot of time and space by writing this distance as:

25×10^{14} miles, or $2.5 \times$ _____

Just a brief note on still other extensions of this system.

Scientists often need to indicate very small numbers — such as a millionth of an inch.

$.000001$ or $\frac{1}{1{,}000{,}000}$

They use negative exponents to indicate tenths, hundredths, etc.

$.1 = 10^{-1}$ $.01 = 10^{-2}$ $.001 = 10^{-3}$

$.0001 =$ _____ $.00001 =$ _____

$.000001 =$ _____

Scientists read 10^{-1}, 10^{-3}, etc., as "ten to the negative first, ten to the negative third," etc.

As you would expect, they extend this as follows:

$.7 = 7 \times .1 = 7 \times 10^{-1}$ $.08 = 8 \times .01 = 8 \times 10^{-2}$

$.012 = 12 \times$ ______ $= 1.2 \times$ ______

$.00091 = 91 \times$ ______ $= 9.1 \times$ ______

To complete the pattern, we agree that any number to the zero power is 1.

$1{,}000{,}000 = 10^6$
$100{,}000 = 10^5$
$10{,}000 =$ ______
$1{,}000 =$ ______
$100 =$ ______
$10 = 10^1$
$1 = 10^0$
$.1 = 10^{-1}$
$.01 =$ ______
$.001 =$ ______
$.0001 =$ ______

The earth is about 93,000,000 miles from the sun, or

9.3×10^7 miles from the sun.

The furthest planet from the sun, Pluto, is about 40 times as far.

$40(9.3 \times 10^7) = 4(9.3 \times 10^8) = 3.72 \times 10^9$

Saturn is only about 10 times as far from the sun as our earth is — or $10 \times 93{,}000{,}000$ miles.

$10(9.3 \times 10^7) =$ _________

Before they used radio telescopes, astronomers could see only about 30,000,000,000 stars, or

$3 \times$ _____ stars.

One of the largest stars they could see has a diameter of 2,400,000,000 miles, or

$2.4 \times$ _____ miles.

Our moon varies in distance from the earth from between 221,000 to 253,000 miles. When it is 250,000 miles away, we express this distance as:

$2.5 \times$ _____ miles.

Uranus, another planet, has four moons. Uranus is 1,800,000,000 miles from our sun, or

_________ miles from the sun.

Here is another scientific problem that requires a shorthand for writing large numbers:

How many atoms are there in the universe? Approximately

300,000,000,000,000,000,000,000,000,000,
000,000,000,000,000,000,000,000,000,000,
000,000,000,000,000 — or

3×10^{74}

Since the word "problem" can be used in many ways, the authors would like to explain what they mean when they use that word.

There are several ways in arithmetic of asking questions in shorthand.

$$\begin{array}{r} 33 \\ -18 \\ \hline \end{array} \qquad \begin{array}{r} 84 \\ +98 \\ \hline \end{array} \qquad \begin{array}{r} 78 \\ \times 5 \\ \hline \end{array} \qquad 13\overline{)234}$$

$35 + 7 = \_\_\_\_$ $\qquad$ $31 + \_\_\_\_ = 50$

$17 \times 9 = \_\_\_\_$ $\qquad$ $\_\_\_\_ \div 8 = 112$

$15 \times \_\_\_\_ = 240$

Perhaps these questions or examples were, once upon a time, problems for you — and it would take lots of time to find correct answers. Now you can probably finish them in a few minutes — most of them without even using scratch paper. Try them.

But we consider those as nothing more than questions you need to be able to answer quickly . . . and when you're finished, you don't feel that you've learned a thing as a result of the effort.

Problems are not like that.

As you answered the first question — "33 − 18 = ?" — did you notice that both 33 and 18 are divisible by 3, and that the answer is, too.

Is the difference between any two multiples of 3 itself a multiple of 3?

That's more like the beginning of what we shall call a problem.

Let's try a few examples of finding the difference between multiples of 3:

$$\begin{array}{r} 30 \\ -12 \\ \hline \end{array} \qquad \begin{array}{r} 42 \\ -27 \\ \hline \end{array} \qquad \begin{array}{r} 75 \\ -36 \\ \hline \end{array} \qquad \begin{array}{r} 132 \\ -51 \\ \hline \end{array}$$

Try other examples that you make up yourself. Can you say confidently that if a multiple of 3 is subtracted from a multiple of 3, the result is a multiple of 3? ______

Is there anything unusual about multiples of 3? Will multiples of 7 behave in much the same way?

$$\begin{array}{r} 21 \\ -14 \\ \hline \end{array} \qquad \begin{array}{r} 70 \\ -42 \\ \hline \end{array} \qquad \begin{array}{r} 91 \\ - \\ \hline \end{array} \qquad \begin{array}{r} \\ - \\ \hline \end{array}$$

Would experiments with multiples of 5, 9, 12, 100, etc., strengthen your conclusion?

Does this same idea hold as well in addition, multiplication, and division as it does in subtraction? ______ (Yes or No)

Now you have an illustration of what the authors call a problem. Work at it. It's not a very hard problem. You ought to settle it by a few minutes of experimenting. But even this simple problem may have a little surprising twist before you're done.

Bill had 12 marbles, Alex had 29, and Harvey had 17. How many do they have altogether?

That is nothing but a question. The answer is 58 marbles . . . but that's not a problem!

Here is another statement about boys and marbles:

Barney has some marbles. He has exactly 3 times as many as Al and exactly 4 times as many as Henry. They have less than 100 marbles among them. What can you say that isn't obvious?

That's a problem!

By "obvious," we might be referring to the fact that Barney has more marbles than either Al or Henry.

(continued on page 14)

It's a little less obvious that Al has more than Henry does, and that Barney has more than Al and Henry put together.

But let's look more deeply into the situation. Could Barney have 7 marbles? . . . 11 marbles? . . . 18 marbles? What is the smallest number of marbles Barney could have? ______. In that case, how many would the boys have altogether? ______

What is the largest number of marbles Barney could have? ______

So, Barney must have ______ or ______ or ______ or ______ or ______ marbles.

Together the three boys must have ______ or ______ or ______ or ______ or ______ marbles.

That was not at all obvious at the outset. We have shed considerable light on the problem.

A Very Famous Problem

Can every even number 4 or larger be obtained by adding two prime numbers?

To understand this problem, we need to know what we mean by *every even number 4 or larger*. We can begin the list:

4, 6, 8, ___, ___, ___, ___, ___, etc.

and we can extend the list as far as we would care to.

We also need to know what *prime numbers* are. Each is larger than 1 and has exactly two different factors — itself and 1. Let's list all that are less than 100, in order:

2, 3, 5, 7, 11, 13, ___, ___, ___

___, ___, ___, ___, ___, ___, ___, ___, ___

___, ___, ___, ___, ___, ___ and ___

Next we set out to obtain numbers in the first list by adding two prime numbers (alike or different) from the second list:

4 =	2 + 2	28 =	+
6 =	3 + 3	30 =	+
8 =	3 + 5	32 =	+
10 =	+	34 =	+
12 =	+	36 =	+
14 =	+	38 =	+
16 =	+	40 =	+
18 =	+	42 =	+
20 =	+	44 =	+
22 =	+	46 =	+
24 =	+	48 =	+
26 =	+	50 =	+

etc.

You may wish to extend the investigation further — to 100 — to 500 — as far as you like.

But, remember — we said it was a famous problem, and it's famous because no one knows the answer. No one has ever found an even number 4 or larger that couldn't be obtained by adding two primes. But not even the finest mathematicians have been able to prove that such an even number does not exist.

If you join the search, you will need lots of scratch paper.

One Problem Leads to Another

Such a good problem almost always suggests other lines of investigation.

As you were working, you probably noticed that some even numbers can be obtained by adding two prime numbers in only one way, and some in more than one way.

$$(8 = 3 + 5 \text{ but } 20 = 3 + 17 = 7 + 13)$$

For each even number, we wish to find all possible pairs whose sum is the even number.

Let's start at 12.

All pairs of primes:

12 = 5 + ____
14 = 7 + ____ = ____ + ____
16 = ____ + ____ = ____ + ____
18 = ____ + ____ = ____ + ____
20 = ____ + ____ = ____ + ____
22 = ____ + ____ = ____ + ____ = ____ + ____
24 = ____ + ____ = ____ + ____ = ____ + ____
26 = ____ + ____ = ____ + ____ = ____ + ____

You may think you see a pattern developing, but:

28 = ____ + ____ = ____ + ____
30 = ____ + ____ = ____ + ____ = ____ + ____
32 = ____ + ____ = ____ + ____
34 = ____ + ____ = ____ + ____ = ____ + ____ = ____ + ____

We skip a bit, and find only two ways to obtain 68:

68 = ____ + ____ = ____ + ____

and nine ways to obtain 90:

90 = ____ + ____ = ____ + ____ = ____ + ____
= ____ + ____ = ____ + ____ = ____ + ____
= ____ + ____ = ____ + ____ = ____ + ____

Perhaps you might be interested in exploring the number of ways other even numbers less than 100 can be obtained by adding two primes. Do you think the nine ways for 90 is a record? It is, but 84 and 86 are close seconds. There are eight ways to obtain each.

Another Problem

The authors have worked out all the pairs whose sums are even numbers up to and including 100.

We can report that all even numbers larger than 12 and through 100 can be obtained by adding two primes in at least 2 different ways.

We wonder: Is there an even number greater than 12 that can be obtained by adding two primes in only 1 way?

All we know is that if there is such a number, it must be larger than 100.

Our records also show that all even numbers 70 or larger can be obtained in at least 4 different ways by adding two prime numbers. The numbers 80, 88, and 98 just barely make the grade, so you might turn one up soon after 100.

What is the smallest even number larger than 100 that cannot be obtained by adding two primes in at least 4 different ways?

The chances of finding one seem encouraging when you realize that 68 turned up suddenly as a number for which there were only 2 ways of adding two primes.

Further, as we reported, our study shows that 90 can be obtained by adding two primes in 9 different ways. That's the record for even numbers 100 or less.

What is the first even number larger than 100 that ties the record, and what is the first even number that beats that record?

We hope you now know what the authors mean when they speak of problems. A good problem requires a lot of thinking and hard work. We hope you will work through some problems with us — and we hope you will tackle some on your own.

In the next chapter, we shall outline many problems. We hope you will pick the one you like best — and begin working at it. You may wish to expand the one you select, or take on others that interest you.

We have a suggestion for you as you start this chapter.

We have outlined many problems. It is our hope that you will find some that interest you particularly.

You will probably have time to work at only a few of these problems, so select carefully.

Read through the chapter as rapidly as you can. Don't try any of the examples. Such a skimming will give you an idea about its contents.

Then read through it again with pencil and paper in hand. Spend a few minutes working at any problems that interest you at all. Try your hand at all of them — just to get a little feeling about each of the problems.

Finally, select your first problem, and begin working at it.

Problem I

Bill said he was looking for combinations of 50 common U.S. coins with a total value of $5.00 — without using dimes — using only silver dollars, halves, quarters, nickels, and pennies. He found one:

6 halves..................	$3.00
6 quarters................	1.50
3 nickels.................	.15
35 pennies................	.35
____ coins	$_____

Bill said that was the only combination he could find.

Can you find other combinations? If so, how many others can you find? If you can't find others, can you give a reason to believe that there are no others?

Problem II*a*

Can you arrange the 10 digits — 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 — to form whole numbers whose sum is exactly 100, using each digit once and only once?

Here is an attempt that's 1 short:

$$39 + 8 + 27 + 10 + 4 + 6 + 5 = 99$$

Here's another:

$$0 + 8 + 16 + 25 + 7 + 31 + 4 + 9 = 100$$

but the digit 1 was used twice.

Keep records of your attempts. Which numbers are easy to obtain as sums?

As you begin, remember that every digit must be used *once and only once.*

And if you get the feeling it is impossible to obtain 100 then try to find a reasonable argument to convince yourself and others you are right.

Problem II*b*

Perhaps you would like to try an easier version of the problem. Use every digit once and only once, but you may use subtraction as well as addition and you must keep the digits in order.

Here are two successful attempts:

$$98 + 7 + 6 - 5 - 4 - 3 + 2 - 1 + 0 = 100$$

$$0 + 1 + 2 + 34 - 5 + 67 - 8 + 9 = 100$$

How many more can you find?

Problem II*c*

Or, if you are having a hard time, remove the requirement that the order be kept:

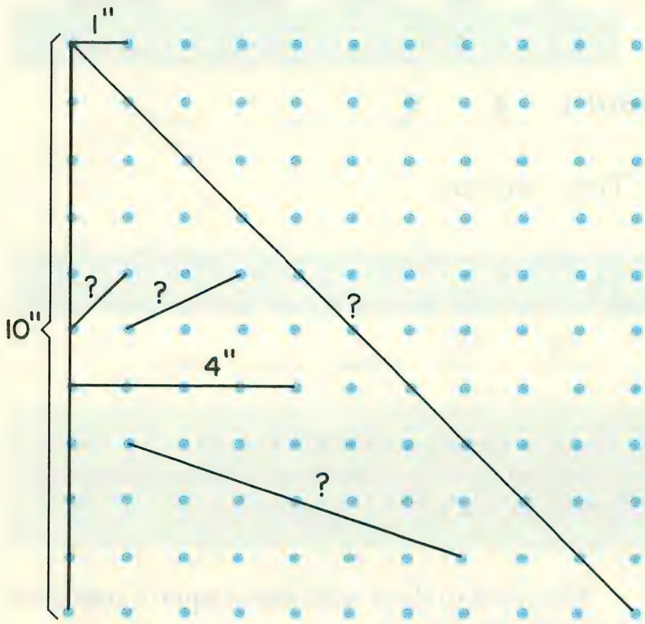
$$68 - 27 + 59 + 0 - 1 - 3 + 4 = 100$$

Can you find at least 5 different ways?

Problem III

On a piece of graph paper, locate points one inch apart in an 11-by-11 array.

Measuring with a ruler, how many different lengths of line segments can you draw that connect two points?



The shortest line segment will be 1″ long. The longest line segment will be a diagonal of the array — a little more than $14\frac{1}{8}$ inches.

You will need to plan your work ahead. How will you know when you've got all of the different lengths? Perhaps it will help to have several 11-by-11 arrays so you won't have too many segments on each drawing.

Would you like to guess before you begin about how many can be found?

Problem IV

Try to find a number which when multiplied by itself gives 2.

We start by knowing that no whole number will work.

1 x 1 = 1 (too small)
2 x 2 = 4 (too large)

So, our number is some place between 1 and 2. We guess at 1.3 and go to work.

1.3 x 1.3 = 1.69 (too small)
1.4 x 1.4 = 1.96 (too small)
1.5 x 1.5 = _____ (too _____)

Now we know that the number is some place between ______ and ______. Keep closing in from both sides, step by step, decimal place by decimal place. How far must you go? Do you think it is possible?

Next, try to find a number which when multiplied by itself gives 3. Then try 4, 5, 6, 7, 8, 9, and 10. What are your conclusions?

Problem V

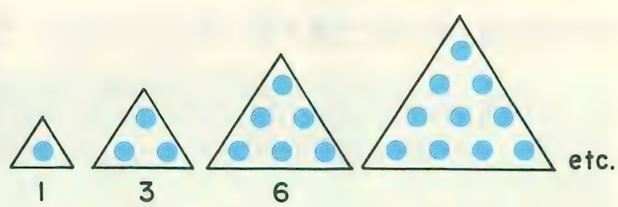
Make a chart showing which letters of the alphabet are used most often and which least often.

Consider noting the frequency of all letters in a newspaper story, the Gettysburg Address, the Pledge of Allegiance to the Flag, a page in a history book — any reading matter available.

If you enjoy this kind of problem, you may be interested in reading *Goldbug* by Edgar Allan Poe. The code for a map of buried treasure is broken by using information you will have uncovered.

Problem VI

You may have already met a sequence of numbers called *triangular numbers.*



You might go on drawing sketches to extend

(continued on page 18)

this list — or you can extend the following:

$$1 = 1$$
$$3 = 1 + 2$$
$$6 = 1 + 2 + 3$$
$$10 = 1 + 2 + 3 + 4$$

Or, extend this sequence, noting the differences between each two terms:

1, 3, 6, 10, ____, ____, ____, ____ etc.

2 3 4 ____ ____ ____ ____

Mary claims that she can obtain every whole number through 50 by adding not more than three of these triangular numbers. (A number may be repeated as one of the three or fewer triangular numbers.)

Do you agree with Mary?

She starts her list in this way:

1 = 1	7 = ___ + ___
2 = 1 + 1	8 = ___ + ___ + ___
3 = 3	9 = ___ + ___
4 = 3 + 1	10 = ___
5 = 3 + 1 + 1	11 = ___ + ___
6 = 6	12 = ___ + ___

Do you believe Mary can keep this up all the way to 50?

How far beyond 50 can you go with this list?

Can some numbers be obtained in more than one way by adding three or fewer triangular numbers?

Yes. For example:

$$30 = 10 + 10 + 10 = 15 + 15 = 21 + 6 + 3 = 28 + 1 + 1$$

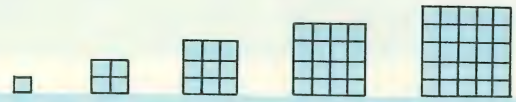
Which numbers can be obtained in only 1 way? Which can be obtained in 2, 3, 4, or more ways?

Investigate these or your own questions. Keep notes of interesting information you uncover.

Problem VII

Mary had another claim.

You are familiar with *square numbers*. They are:



1, 4, ____, ____, ____

(diff.) 3 5 ____ ____

They continue:

25, ____, ____, ____, ____, ____ etc.

11 13 ____ ____ ____

Mary's second claim is that she can obtain every whole number less than 50 by adding not more than four square numbers.

She tried to do it with three square numbers, but stumbled on 7:

7 = ___ + ___ + ___

and again on 15:

15 = ___ + ___ + ___

At 28, she noted that she needed four again, but found 3 different ways to do it:

28 = ___ + ___ + ___ + ___ = ___ + ___ + ___ + ___

= ___ + ___ + ___ + ___

When asked why she didn't carry her study beyond 50, she said that she was more interested in another problem involving square numbers.

What interesting developments can you find in checking Mary's claim and extending it to 100?

What numbers more than 20 and less than 100 can be obtained in only one way as the sum of four or fewer square numbers?

Problem VIII

Mary explained her new problem in this way:

"I'm looking for a square number that is exactly twice as large as another square number.

"I couldn't find any square number less than 100 exactly twice as large as another less than 100 — but I came close:

<u>49</u> is almost twice <u>25</u>

"So, I'm working on square numbers larger than 100. I've found 144 (12×12) and 289 (17×17). That's close. So, I'm still working!"

Maybe you would like to join in the search.

Problem IX

But Mary hadn't finished.

"I'm also looking for a square number that is 3 times as large as another. Again, it's easy to come close, but hard to be exact.

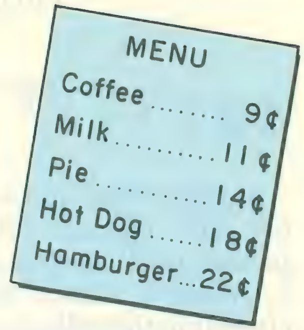
<u>49</u> is just 1 more than 3 x <u>16</u>

"I'm carrying on these two searches at once."

Perhaps you would like to join the search. You might even broaden it — searching the list of square numbers for couples in which one is 2, 3, 4, 5, 6, 7, 8, 9, or 10 times the other.

In thinking about any problem or working at it, keep your eyes open for interesting questions of your own that you would like to explore. Very often such investigations are by far the most interesting.

Problem X



A snack bar had this menu.

At first glance, there seems to be little of interest in those prices.

However, they must have been carefully selected!

What is unusual about those particular prices?

Let's suppose that no customer bought more than one of a given item at a time, but that he could buy one or all five or any combination of items.

A group of 6 boys bought snacks. Each paid his own check. Here are the amounts they paid. (There was no tax.)

Al – 27¢	Chet – 34¢	Ed – 52¢
Bill – 42¢	Don – 32¢	Fred – 54¢

Think about that information and study the menu.

List different combinations of items each of the 6 boys might have had.

Al might have had coffee and a hot dog — $9¢ + 18¢ = 27¢$. What other combination adds to 27¢?

What about the others?

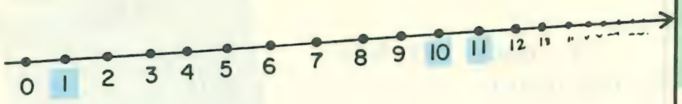
What combination of items could amount to 30¢, 35¢, 44¢, 50¢?

How many different combinations of items are there?

What is unusual about the prices on this menu?

Could you work out another menu that is unusual in the same way? Start with 3 items; then 4 items; then 5 items. You may wish to try for a much longer list that is unusual.

Problem XI



Imagine that you walked out along a number line labeled with whole numbers and picked up all those made only of the digits 0 or 1. While 0 is itself such a label, let's not include it in the collection.

Thus, the first one you would pick up is 1, the next is 10, the next 11, etc. Let's list the first 15 you would pick up:

1	1st		6th		11th
10	2nd		7th		12th
11	3rd		8th		13th
	4th		9th		14th
	5th		10th		15th

(Make long columns rather than the short ones we made above.)

Find a pattern that will help you know without looking at the list which will be the 50th label, the 71st label, the 99th, the 150th, and any label anyone would ask for.

110,010	50th	1,100,011	99th
1,000,111	71th	10,010,110	150th

As a hint for your search for such a pattern, make a list of these labels:

1st	2nd	4th	8th	16th	32th	etc.
1	10	100				

Problem XII

Try the same basic problem, but pick up all labels after 0 that contain no digits other than 0, 1, 2, 3, and 4.

1	1st	4	4th	12	7th		10th
2	2nd	10	5th		8th		11th
3	3rd	11	6th		9th		12th

Again, find a pattern that would help you know without checking your list what the 78th, 215th, etc., labels would be.

Make a similar study of the digits 0 through 7; then a study of 0 through 9.

Problem XIII

Palindromic numbers do not change if their digits are reversed. Examples are:

11, 22, 101, 383, 1771, 81618

The number 78 is not palindromic. Reversing digits changes it to 87. If these two numbers are added:

$$\begin{array}{r} 78 \\ +87 \\ \hline 165 \end{array} \quad \text{a reversal of digits}$$

the sum is not palindromic because 165 and 561 are different. But let's add them and, if the sum is not palindromic, we'll reverse its digits and add . . . until we reach one of these numbers.

$$\begin{array}{rl} 78 & \\ +87 & \text{reverse digits and add} \\ \hline 165 & \\ 561 & \text{reverse digits and add} \\ \hline 726 & \\ 627 & \text{reverse digits and add} \\ \hline 1353 & \\ 3531 & \text{reverse digits and add} \\ \hline 4884 & \leftarrow \text{It's palindromic!} \end{array}$$

Here are a few short ones for you to try:

$$\begin{array}{r} 67 \\ +76 \\ \hline 143 \\ \\ \hline \end{array} \qquad \begin{array}{r} 59 \\ +95 \\ \hline 154 \\ \\ \hline \\ \\ \hline \end{array} \qquad \begin{array}{r} 38 \\ +83 \\ \hline \end{array} \qquad \begin{array}{r} 27 \\ +72 \\ \hline \end{array}$$

Few numbers less than 100 are interesting when we repeat the digit reversing and adding. However, 97 does not give up easily.

```
 97
+79
----
+
----
+
----
+
----
+
----
+
----
```

6 reversals and 6 additions are required

← It's palindromic.

Can you find any number less than 100 that would require more than 6 reversals and 6 additions?

Try some numbers between 100 and 200. For example, try 168, 157, 175, 188, 198.

Can you find a number that does not become palindromic no matter how many digit reversals and additions you carry out?

Make notes of interesting examples you try.

Problem XIV

What is the largest number that can be obtained as the product of two or more numbers whose sum is given?

For example, suppose 23 is given as the sum. We can begin by finding several combinations of numbers whose sum is 23, and finding the products of those same numbers.

$11 + 12 = 23$ and $11 \times 12 =$ ____

$3 + 20 = 23$ and $3 \times 20 =$ ____

$10 + 10 + 3 = 23$ and $10 \times 10 \times 3 =$ ____

$8 + 8 + 7 = 23$ and $8 \times 8 \times 7 =$ ____

$2 + 6 + 8 + 7 = 23$ and $2 \times 6 \times 8 \times 7 =$ ____

Keep trying combinations of numbers whose sum is 23 until you have the largest product possible. (The largest product is 3,072.)

You will be surprised! Did the fact that we began with 23 as the sum lead to that surprising result?

To find the answer to that, try the same investigation with 17 in place of 23; and with 46 . . . or with any number that comes to mind.

Report your results.

Consider working on the next problem. It is a variation on this one.

Problem XV

This problem is slightly different from the last. The combination of numbers whose sum is a given number must all be different.

What is the largest number that can be obtained as the product of *different* factors whose sum is given?

Again, begin by considering 23 as the sum.

$11 + 12 = 23$ and $11 \times 12 =$ ____

$6 + 8 + 9 = 23$ and $6 \times 8 \times 9 =$ ____

$4 + 5 + 6 + 8 = 23$ and $4 \times 5 \times 6 \times 8 =$ ____

. . . and keep going until you are able to find a combination of different numbers whose sum is 23 and whose product is 1,260.

Try other numbers (in place of 23) as the sum. Begin with small numbers.

$1 + 4 = 5$ and $1 \times 4 = 4$

$2 + 3 = 5$ and $2 \times 3 = 6$

There is no other combination to try because $1 + 1 + 3$ and $1 + 2 + 2$ break the rule that all must be different.

Try 6, then 7, then 8, etc., until you notice a pattern. Test your pattern with 15 or 27 or some other number.

Can you develop a system so that you can quickly find the best combination as soon as you know the sum?

If you are enjoying the investigation, try the next which is based on another change in the rules.

Problem XVI

What is the largest number that can be obtained as the product of different prime numbers when the sum of the primes is given?

In this variation, we are limited to primes as factors and it is required that they be different.

If, again, we start with 23, we don't have many combinations of primes whose sum is 23. There are a total of only three:

3 + 7 + 13 = 23 and 3 x 7 x 13 = _____

5 + 7 + 11 = 23 and 5 x 7 x 11 = _____

2 + 3 + 7 + 11 = 23 and 2 x 3 x 7 x 11 = _____

Let's try another number, such as 30. There are three combinations of two primes whose sum is 30.

7 + 23 = 30 and 7 x 23 = _____

11 + 19 = 30 and 11 x 19 = _____

13 + 17 = 30 and 13 x 17 = _____

How many combinations of three primes have 30 as sum? And why must 2 be one of the primes?

Can you show that there is no combination of four primes whose sum is 30?

There is a combination of five primes whose sum is 30 and whose product is 2730. What are the five prime numbers?

___ x ___ x ___ x ___ x ___ = 2730

Problem XVII

This problem is related to the three previous problems, but *sum* and *product* change places.

What is the smallest sum of factors whose product is given?

Suppose the given product is 36. Since the factors need not all be different, there are quite a few.

18 x 2 = 36 and 18 + 2 = _____

12 x 3 = 36 and 12 + 3 = _____

9 x 4 = 36 and 9 + 4 = _____

6 x 6 = 36 and 6 + 6 _____

Since we are looking for the smallest sum, we are moving in the right direction. Let's use more factors.

9 x 2 x 2 = 36 and 9 + 2 + 2 = _____

6 x 2 x 3 = 36 and 6 + 2 + 3 = _____

When you have obtained a sum as small as 10, you will have found the smallest sum of factors whose product is 36.

Try other numbers that have several combinations of factors such as 12, 18, 24, 30, 42, 56, etc. Include 64 and 81 as interesting examples. Find other interesting examples.

In each study, what do you notice about all the factors in the combination having the smallest sum?

Can you find an example in which there are two different combinations having the smallest sum — provided that 4 is always written as 2×2?

The number 4 is a little peculiar. It is one of the two numbers which is both the sum and product of a pair of numbers. What is the other number?

4 = ___ + ___ = ___ x ___

___ = ___ + ___ = ___ x ___

Problem XVIII

Can you write each whole number from 0 through 20 with four 4's?

You may use addition, subtraction, multiplication, division, or any combination of these. You may use decimal points and symbols of grouping. But it must be with exactly four 4's — no more, no less.

Suppose that we tackle a similar problem — writing numbers with four 5's. Here are a few:

$0 = 55 - 55$

$1 = 55 \div 55$

$2 = (5 \div 5) + (5 \div 5)$

$3 = (5 + 5 + 5) \div 5$

$4 = [(5 \times 5) - 5] \div 5$

$5 = (5 \times .5) + (5 \times .5)$

$6 = (55 \div 5) - 5$

$7 = 5 + [(5 + 5) \div 5]$

$8 = 5 + .5 + (.5 \times 5)$

$9 = 5 + 5 - .5 - .5$

$10 = 5 + [5 \times (5 \div 5)]$

Of course, someone else might have an entirely different list:

$0 = 55 \times (5 - 5)$

$1 = 5 - 5 + .5 + .5$

$2 = .5 + .5 + .5 + .5$

$3 = 5 - [(5 \times .5) - .5]$

$4 = [(5 - .5) \div .5] - 5$

$5 = [(5 - 5) \times 5] + 5$

Now that you know the rules, you can begin writing as many numbers as you can with four 4's. Try for all numbers 0 through 20 — and as many beyond 20 as you can find.

Next, try four 6's; then on to four 7's. Perhaps you can get a contest underway. Get your parents and friends to join in the search.

Problem XIX

Here is a slight variation of the last problem:

Write each whole number from 0 through 20 using any group of four digits — such as 1, 9, 6, 8 — in place of the four 4's.

$0 = (8 + 1 - 9) \times 6$

$9 = 8 - (6.1 + .9)$

$2 = 8 - [6 \times (.1 + .9)]$

$3 = 18 - 9 - 6$

$4 = 8 + 6 - 9 - 1$

$5 = 19 - 8 - 6$

$6 = 6 \times 1 \times (9 - 8)$

$7 = (6 \times 1) + (9 - 8)$

Problem XX

Another variation of the problem is the telephone number problem:

Select any telephone number. Find as many ways as you can to arrange the 3 area code digits, the 3 exchange digits, and the remaining 4 digits so that all arrangements give the same number.

Suppose that we have the telephone number:

408 – 624 – 8115

(0)...... $48 \times 0 = 6 - 2 - 4 = 85 \times (1 - 1)$

(2)...... $8.0 \div 4 = 2 \times (.4 + .6) = 8 + 5 - 11$

(4)...... $8 - 4 + 0 = 24 \div 6 = 8 - 5 + (1 \times 1)$

(5)..... $40 \div 8 = (6 + 4) \div 2 = 5 + [8 \times (1 - 1)]$

(8)...... $8 + (4 \times 0) = 6 + 4 - 2 = 11 + 5 - 8$

(12)..... $4 + 8 + 0 = 6 + 2 + 4 = 8 + 5 - (1 \times 1)$

(20).... $80 \div 4 = (6 + 4) \times 2 = (15 + 1) \div .8$

(32) $8 \times 4.0 = 64 \div 2 = [8 \times (1 + 1)] \div .5$

(48).... $48 + 0 = 6 \times 2 \times 4 = 8 \times [5 + (1 \times 1)]$

You might find arrangements which give decimal fractions:

(8.4).... $8.0 + .4 = 2.4 + 6 = 8.5 - (.1 \times 1)$

and common fractions:

$\frac{1}{2}$ ...... $\frac{4 + 0}{8} = \frac{4}{6 + 2} = \frac{5 - 1}{8 \times 1}$

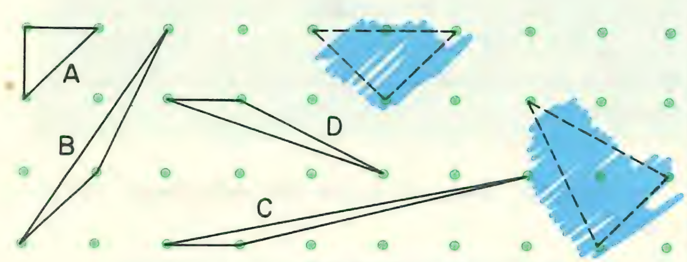
You might make up a game based on the idea above that uses an automobile license plate, a Social Security number, the first two and last three digits of your zip code. Games that you invent are the ones that are the most fun.

Select a Problem

Select the problem you wish to tackle first, and plunge into it. Please share all the interesting facts you uncover with others. Perhaps you can organize your work as a Science Fair project.

A Problem from Geometry or, Doodler's Delight

What would you make of the following bit of doodling?



3d (0)

All are triangles, but two were erased. Why?

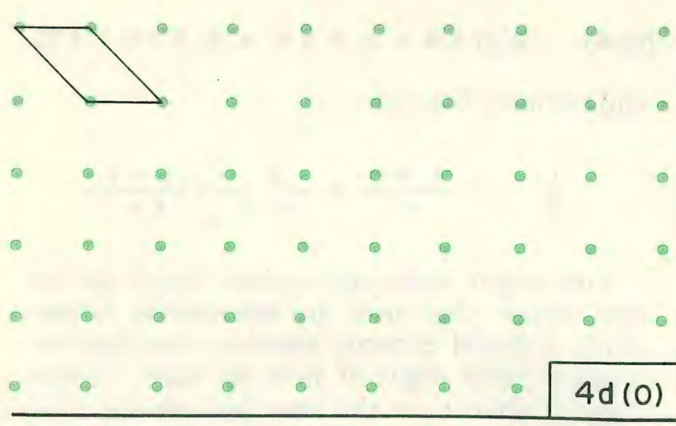
What's the "$3d(0)$"?

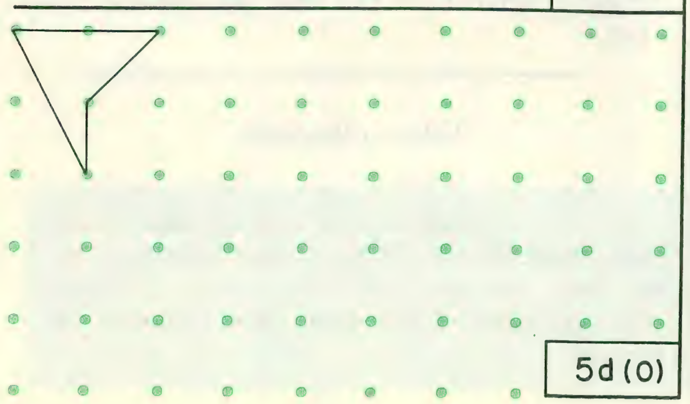
How are A, B, C, and D alike, but different from those erased?

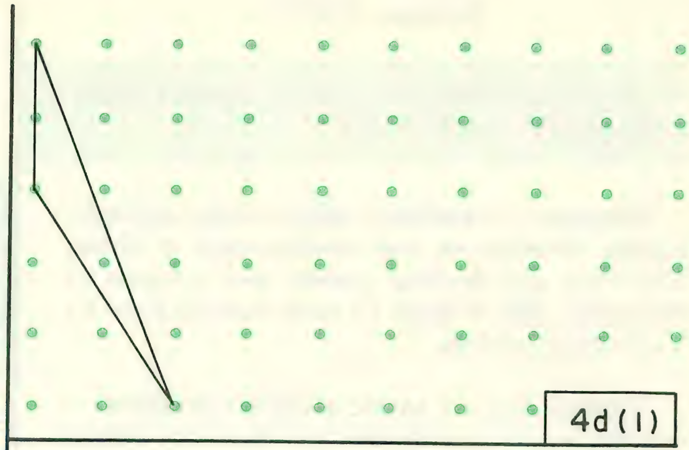
Would you be surprised to find out that "$3d(0)$" is shorthand for "3-dotter with 0 dots inside"?

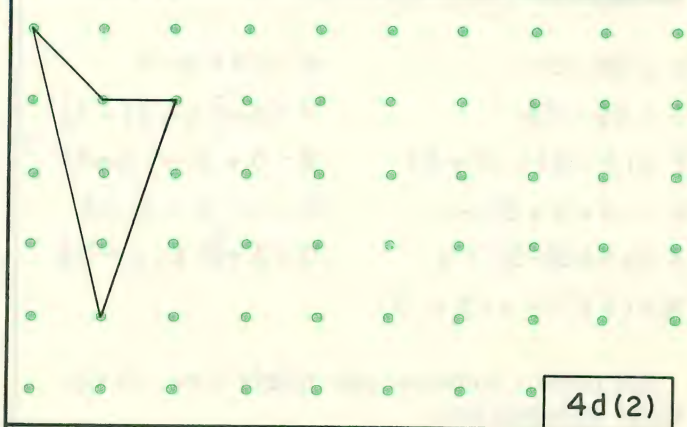
So, one triangle was erased because it touched 4 dots and the other because it had 1 dot inside.

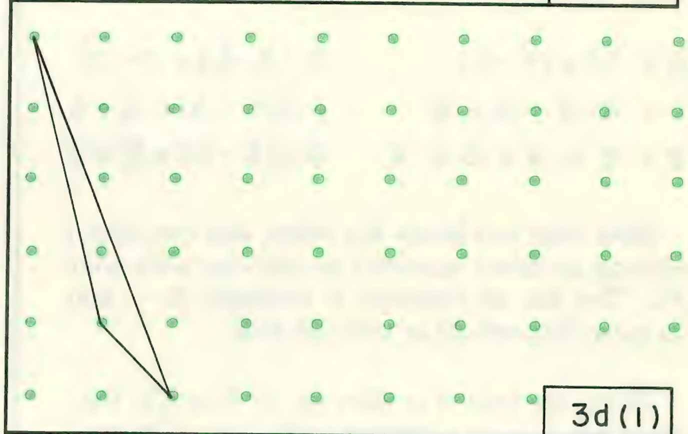
Now it's your turn to doodle. Make several different shapes that belong in each group.







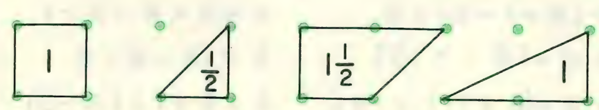




The last doodling — $3d(1)$ — was probably harder than the others . . . but still fun.

Now for a bit of work! (Every good problem is a mixture of fun and work.)

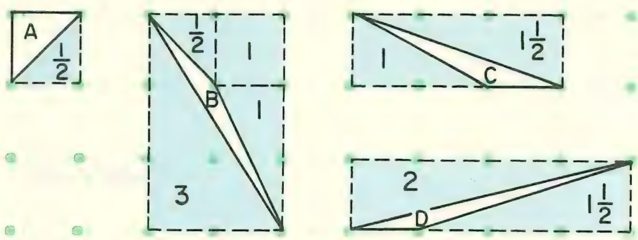
If we consider  as 1 unit of area, what is the area of each of the doodles? Mark the area of each next to it or inside the shape. (Here's some help — and there is more on the next page.)



Surprise!

You may have some trouble finding the area of some of the doodles.

It's helpful to remember that the diagonal of a rectangle divides the rectangle in half.

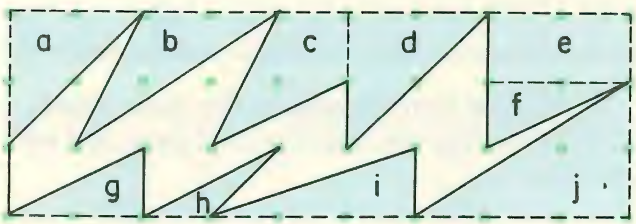


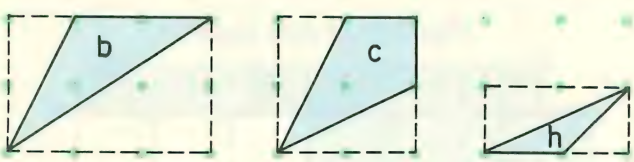
(all numbers refer to units of area)

$A = 1 - \frac{1}{2} = \frac{1}{2}$ $\quad$ $B = 6 - 3 - \frac{1}{2} - 1 - 1 =$ ____

$C = 3 - 1 - 1\frac{1}{2} =$ ____ $\quad$ $D =$ ________ $=$ ____

Can you find the area of this streak of lightning? *





a = 2 $\quad$ b = ____ $\quad$ c = ____ $\quad$ d = ____

e = ____ $\quad$ f = ____ $\quad$ g = ____ $\quad$ h = ____

i = ____ $\quad$ j = ____

a + b + c + d + e + f + g + h + i + j = ____

The area of the big rectangle around the lightning streak is ______ units. The part shaded blue is ______ units.

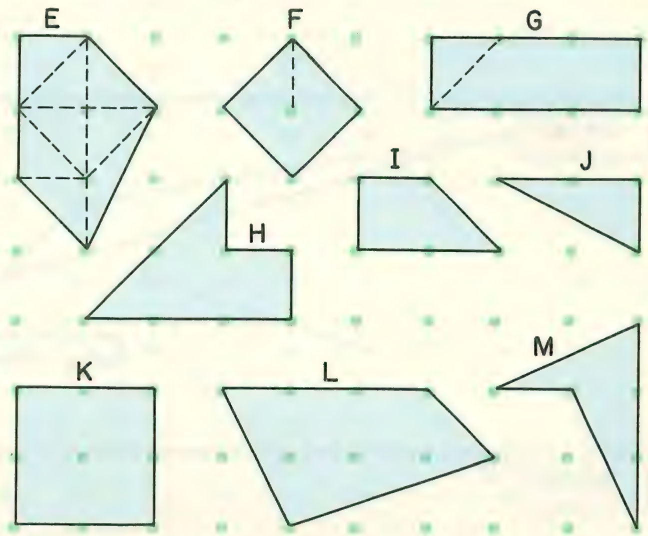
So, the area of the unshaded part must be ______ units.

Now, perhaps, you have some ideas that will help you find the area of any doodle on the previous page.

* The lightning streak is a 20-dotter with 1 dot inside, a $20d(1)$.

Suppose that you tried to draw segments inside a shape to divide it up into 3-dotters with 0 dots inside.

Remember that the segments must connect dots and must not cross another segment except at a dot. The first one is completed. Others may be started. Please complete them all.

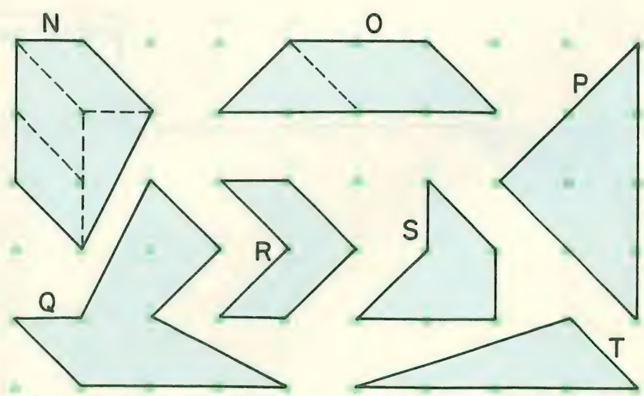


Does this suggest another way to find the areas of the doodles? Remember, all $3d(0)$ have the same area.

Write the areas of the shapes above in this chart:

E	F	G	H	I	J	K	L	M
4								

Or, you might decide to break up shapes into 4-dotters with 0 dots inside, with a $3d(0)$ left over sometimes.

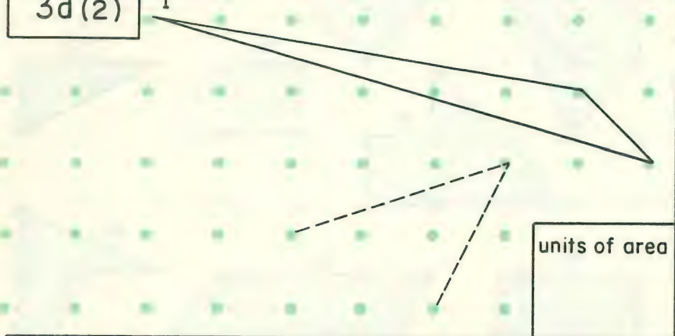


N	O	P	Q	R	S	T
4						

In each section draw at least two of the shapes indicated by our doodler's shorthand. Find the area of each by any method you like.

You probably are already convinced that all shapes described by the same piece of shorthand have the same area.

3d(2) I



units of area

5d(2) II

units of area

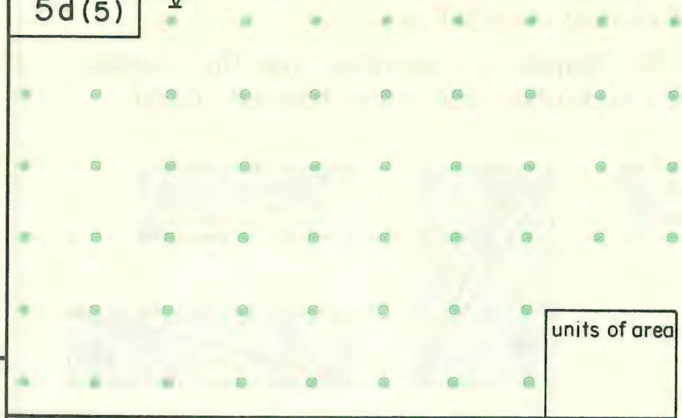
7d(2) III

units of area

7d(4) IV

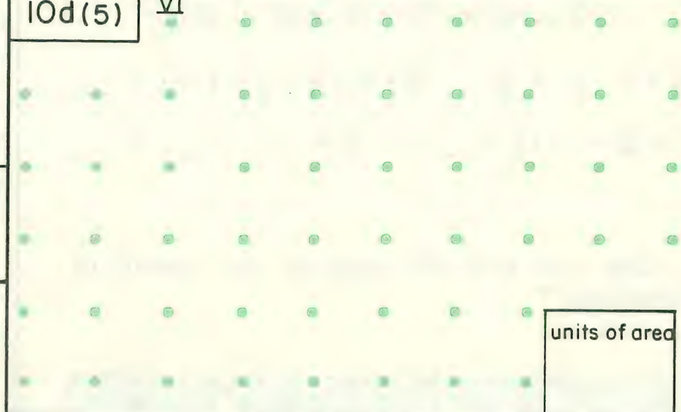
units of area

5d(5) V



units of area

10d(5) VI



units of area

VII

In the blue tinted blocks in the chart below, record the areas you found on this page and on page 24.

Number of dots inside

Number of dots on the boundary	(0)	(1)	(2)	(3)	(4)	(5)
3	$\frac{1}{2}$					
4						
5						
6						
7						
8						
9						
10						

Do you notice a pattern in the entries that suggests a way to complete the chart without further experiments?

If so, take a chance — and then check some of these new entries by experiments. Did the pattern hold up?

If you can find the entry in the first column of the chart on page 26 — under the (0) — the rest is easy. For example:

	(0)	(1)	(2)	(3)	(4)
5	$1\frac{1}{2}$	$1\frac{1}{2}+1$	$1\frac{1}{2}+2$	$1\frac{1}{2}+$	$1\frac{1}{2}+$

So, let's turn to that column of results for shapes with 0 dots inside.

If we have 1 dot or 2 dots, the area is 0.

After the second dot, each additional dot leads to the addition of $\frac{1}{2}$ unit of area.

This suggests a way to find any entry in the left-hand column.

Subtract 2 from the number of dots on the boundary and divide the result by 2.

If we use b as a placeholder for the number of dots on the boundary, we can express the idea by writing:

$$[(b-2)\div 2]$$

To move across the chart, we know we must add as many units of area as there are dots inside. If we use i as a placeholder for the number of dots inside, we can express that idea by writing:

$$+\ i$$

And putting them together, we can find any entry in the chart:

$$[(b-2)\div 2]+i = \text{units of area}$$

That one expression is a summary of all our investigations of this problem. It is a way of describing all the entries in our chart and in extensions of that chart.

If you know the number of dots on the boundary and the number of dots inside, you can find the area without even looking at the shape.

The "streak of lightning" on page 25 is a $20d(1)$. What is its area?

$$[(20-2)\div 2]+ \_\_\_\_ = \_\_\_\_$$

Check your answer on page 25.

Complete the following statements. Use any method you wish, including making sketches.

a 4d (2) has ____ units of area
a 7d (1) has ____ units of area
a 5d (7) has ____ units of area
a 20d (13) has ____ units of area
a 3d ( ) has $3\frac{1}{2}$ units of area
a 6d ( ) has 5 units of area
a 12d ( ) has 9 units of area
a 13d ( ) has $9\frac{1}{2}$ units of area
a 4d ( ) has 17 units of area
a ___d (3) has $3\frac{1}{2}$ units of area
a ___d (2) has $4\frac{1}{2}$ units of area
a ___d (8) has 9 units of area
a ___d (5) has $9\frac{1}{2}$ units of area

Complete the following in four different ways:

a ___d ( ) has $8\frac{1}{2}$ units of area
a ___d ( ) has $8\frac{1}{2}$ units of area
a ___d ( ) has $8\frac{1}{2}$ units of area
a ___d ( ) has $8\frac{1}{2}$ units of area

In the late 1800's, a mathematician discovered the formula you have worked out. His name was Pick — and the formula is called *Pick's theorem.*

Chapter FIVE — A Problem from Arithmetic

Legend has it that the Grand Vizier invented the game of chess for King Shirham of India. After learning the rules, the King was fascinated by the game and wished to reward the Grand Vizier (who was also a mathematician).

"Oh King," said the inventor, "give me two grains of wheat to put on one square of the chessboard, four for another square, eight for another, sixteen for another — and keep doubling the number of grains until we have accounted for the 64 squares on the board.

"That is a modest request," the King replied. "It is granted!" And he ordered his servants to bring in a bag of wheat.

The legend is told in many different ways. However, you can be sure that in no version is the King able to fulfill his promise.

You might guess that a chart something like the following was found in the Grand Vizier's notebook.

No. of squares	No. of grains for each square	Cumulative total
1st	2	2
2nd	4	6
3rd	8	14
4th	16	30
5th	32	62
6th	64	
7th	128	
8th		
9th		
10th		
11th		
12th		
13th		
14th	16,384	
15th		
16th		
17th		
18th		

A total of more than half a million grains of wheat — and only 18 of the 64 squares accounted for.

As he continued, the Grand Vizier saw no reason to keep the cumulative total. He had stopped adding the previous amount. (Can you see why?)

	Number of grains for each square
19th	524,288
20th	
21th	
22th	
23th	
24th	
25th	
26th	
27th	
28th	
29th	
30th	

Now the cumulative total has grown to

________________________.

	Number of grains for the square indicated	Cumulative total of grains
64th	18,446,744,073,709,551,616	36,893,488,147,419,103,230

It has been estimated that there are about 5,000,000 grains of wheat in a bushel. The total wheat production of the world in a year is about 2,000,000,000 bushels.

Can you estimate the number of years it would take to fill the Grand Vizier's request if all wheat production were available?

We can only hope that the King and the Grand Vizier had a good laugh and then noticed some interesting facts about all those computations, such as:

1.

1st square	2
2nd square	4
3rd square	8

2.

2nd square	4
4th square	16
6th square	

3.

2nd square	4
3rd square	
5th square	

4.

3rd square	
4th square	
7th square	

5.

2nd	4
2nd	4
4th	

6.

9th	512
1st	
10th	

7.

10th	
3rd	
13th	

8.

12th	
2nd	
14th	

9.

20th	1,048,576
3rd	
23rd	

10.

6th	64
9th	512

11.

5th	
5th	

12.

	512
	4

13.

	64
	64

14.

	32
	256

15.

	128
	4,096

16.

	512
	512

17.

2		5		7
4	x	32	=	

18.

32	x	64	=	

19.

32	x			8,192

20.

8		3		5
256	÷	8	=	

21.

1,024	÷		=	64

22.

16,384	÷	128	=	

23. 16 x ______ = 256

24. 32 x 64 = ______

25. 8,192 ÷ 128 = ______

26. 16 x 1,024 = ______

27. 256 x 256 = ______

28. 16,384 x 32 = ______

Because the Grand Vizier was also a skillful mathematician, there is hope that his interest turned to a study of the fascinating results on the previous page.

Mathematicians have extended the results.

		Shorthand		
2 used as a factor 0 times		2^0	is equal to	1
2 used as a factor 1 time	2	2^1		2
2 used as a factor 2 times	$2 \cdot 2$	2^2		4
2 used as a factor 3 times	$2 \cdot 2 \cdot 2$	2^3		8
2 used as a factor 4 times		2^4		16
2 used as a factor 5 times		2^5		32
2 used as a factor 6 times		2^6		
		2^7		
		2^8		
		2^9		
		2^{10}		

2^1 is read as "2 to the 1st power"

2^2 is read as "2 to the 2nd power" or "2 squared"

2^3 is read as "2 to the 3rd power" or "2 cubed"

2^4 is read as "2 to the 4th power"

2^5 is read as "2 to the 5th power"

. . . . etc.

2^0 is read as "2 to the zeroth power" but what does it mean? It certainly fits the pattern in the "shorthand" column.

In the column on the right, each number is half of the number below it. So, we hope that 2^0 is half of 2, or

$$\boxed{2^0 = 1}\ .$$

The following examples use 2^0 as 1. Does it cause any trouble?

$2^2 \cdot 2^3 = 2^5$ because $4 \times 8 = 32$

$2^1 \cdot 2^6 = 2^7$ because $2 \times 64 =$ ____

$2^0 \cdot 2^3 = 2^3$ because $1 \times 8 =$ ____

$2^4 \cdot 2^3 =$ ____ because __________

$2^2 \cdot 2^4 =$ ____ because __________

$2^5 \cdot 2^0 =$ ____ because __________

$2^0 \cdot 2^0 =$ ____ because __________

Letting $2^0 = 1$ passes every test.

$2^5 \div 2^3 = 2^2$ because $32 \div 8 = 4$

$2^6 \div 2^1 =$ ____ because __________

$2^5 \div$ ____ $= 2^1$ because __________

$2^6 \div 2^0 =$ ____ because __________

____ $\div 2^4 = 2^2$ because __________

$2^8 \cdot 2^5 = 2^{13} = 8{,}192$

$2^{17} \cdot 2^3 =$ ____ $=$ __________

$2^2 \cdot 8 =$ ____ $=$ __________

$256 \cdot 64 =$ ____ $=$ __________

Of course, the Grand Vizier, being a mathematician, would wonder if other tables could be built that would have a similar pattern.

Suppose that we triple each time instead of doubling.

I.

3 as a factor 0 times		3^0	1
3 as a factor 1 time	3	3^1	3
3 as a factor 2 times	$3 \cdot 3$	3^2	9
3 as a factor 3 times	$3 \cdot 3 \cdot 3$	3^3	
3 as a factor 4 times	$3 \cdot 3 \cdot 3 \cdot 3$	3^4	
3 as a factor 5 times			
3 as a factor 6 times			

II.

3^3	27
3^2	9
3^5	

3^2	9
3^1	
3^3	

3^4	
3^0	
3^4	81

	81
	9

	27
	729

	81
	243

III.

Let's extend the table.

3^7	2,187
3^8	
3^{13}	

IV.

3^5	243
3^3	
3^8	

3^6	729
	81

	27
	2,187

	81
	177,147

	729
	729

	243
	1,594,323

V.

5		5		10
243	x	243	=	
2,187	x		=	2,187
19,683	÷	243	=	
6,561	=	531,441	÷	
1,594,323	=	729	x	

VI.

	x	27	=	177,147
	÷	81	=	27

VII.

$243 \overline{)531{,}441}$

$$\begin{array}{r} 729 \\ \times\ 243 \\ \hline \end{array}$$

More tables of powers

I.

	5^0	1
5	5^1	5
5·5	5^2	25
5·5·5	5^3	
	5^4	

II.

	7^0	1
7	7^1	7
7·7	7^2	49
7·7·7	7^3	
	7^4	

III.

	10^0	1
10	10^1	10
10·10	10^2	100

In the examples below, use the tables for shortcuts if you wish to.

IV.

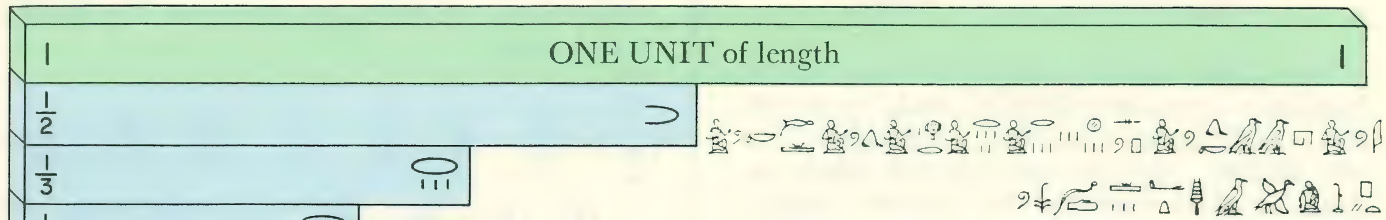
2		3		5
25	x	125	=	
		4		
15,625	÷		=	
390,625	=		x	3,125
		7		3
	=		÷	

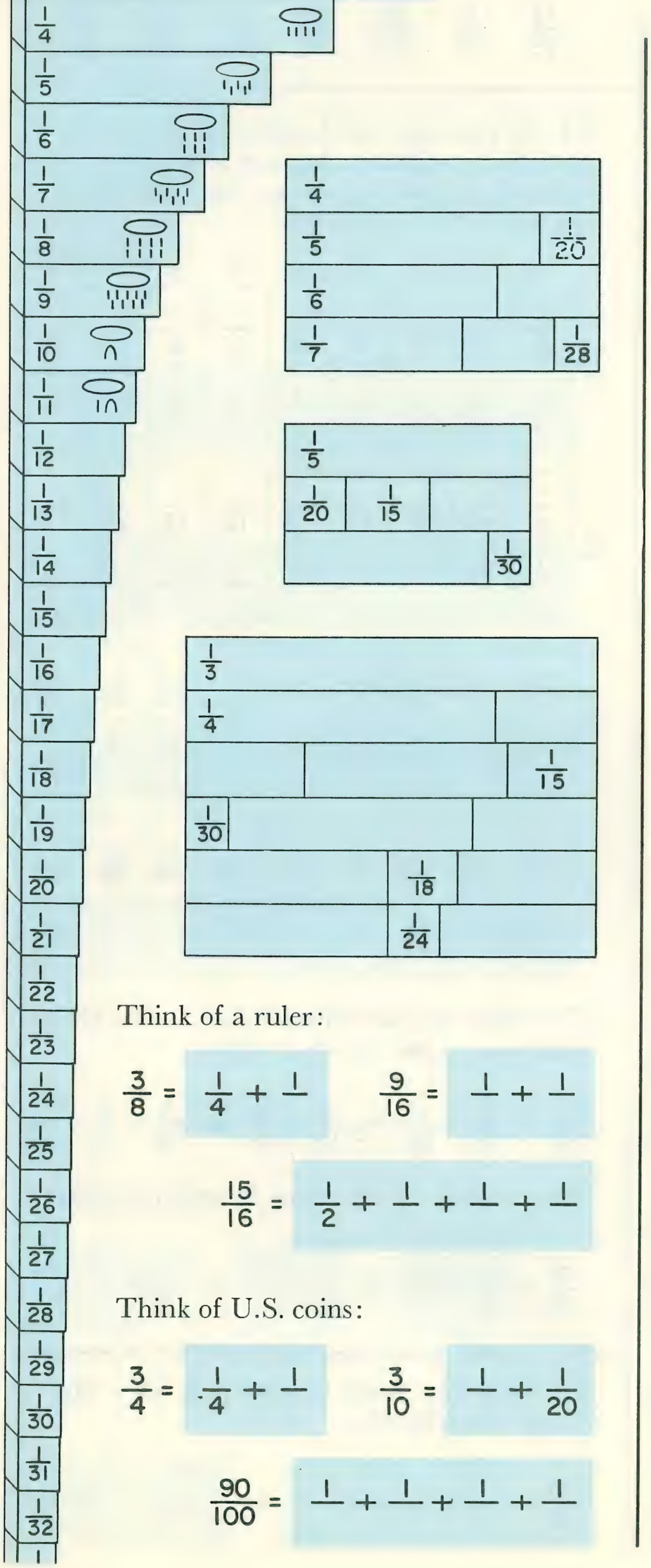
V.

3		3		6
343	x		=	
4				1
	÷		=	
	÷	49	=	16,807
				5
16,807	x		=	

VI.

3		4		1
	=	10,000	÷	10
		2		
1,000,000	÷		=	
0		3		
	x		=	
2				6
	x		=	





Think of a ruler:

$$\frac{3}{8} = \frac{1}{4} + \frac{1}{\quad} \qquad \frac{9}{16} = \frac{1}{\quad} + \frac{1}{\quad}$$

$$\frac{15}{16} = \frac{1}{2} + \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad}$$

Think of U.S. coins:

$$\frac{3}{4} = \frac{1}{4} + \frac{1}{\quad} \qquad \frac{3}{10} = \frac{1}{\quad} + \frac{1}{20}$$

$$\frac{90}{100} = \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad}$$

Translated from Egyptian hieroglyphics, this says:

I am half of my friend who is a third of one.

What is the quantity saying it?

In 1858, a young Scotsman, Mr. A. Henry Rhind, reportedly found a piece of papyrus in a ruins near Thebes. It had originally been about 18 feet long and 13 inches high, but parts were missing.

Surprisingly enough, some of those missing parts turned up at the New York Historical Society in about 1900.

The scroll was written about 1700 B.C. When finally translated, it turned out to be a practical handbook showing how the Egyptians counted, computed, and measured more than 3500 years ago.

They used fractions in measurement, but for some reason they followed peculiar rules. They limited themselves to what we call *unit fractions,* fractions with 1 as the numerator. (There was one exception: they had a way of writing $\frac{2}{3}$, but we shall overlook this exception.) And, they would not repeat a unit fraction in an expression. They would not express $\frac{3}{4}$ by writing:

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

Instead, they would use two different unit fractions and write:

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$$

Thus, they would choose a unit — such as the green bar above — and consider as smaller units only $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc.

Try your hand at some of the problems suggested by the sketches.

Egyptians had fraction troubles.

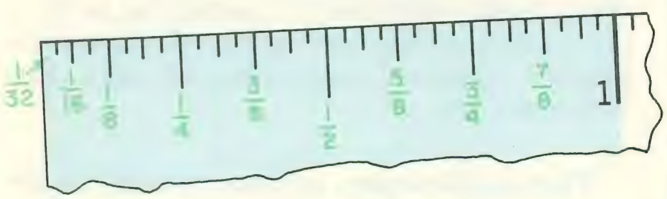
This 3500-year-old way of thinking about fractions is certainly difficult. And, on top of that, the Egyptian method of writing fractions was clumsy. (See the old Egyptian labels on fractions of the unit on the previous page.) The Egyptians would describe $\frac{3}{7}$ by thinking:

$$\frac{3}{7} = \frac{1}{28} + \frac{1}{7} + \frac{1}{4}$$

and they would express $\frac{3}{7}$ this way, without addition signs:

I. Let's see if we can overcome some of these problems, without getting involved in their clumsy notation.

If you think about the divisions on a ruler, you won't have much trouble with the examples below. Through a magnifying glass, an inch on a ruler looks like this:



a.	$\frac{5}{8}$	$\frac{1}{2} + \frac{}{8}$
b.	$\frac{3}{8}$	$\frac{1}{} + \frac{1}{}$
c.	$\frac{9}{16}$	$\frac{1}{2} + \frac{1}{}$
d.	—	$\frac{1}{2} + \frac{1}{4}$
e.	$\frac{7}{8}$	$\frac{1}{2} + \frac{1}{} + \frac{1}{}$
f.	$\frac{3}{16}$	$\frac{1}{} + \frac{1}{}$
g.	$\frac{7}{16}$	$\frac{1}{} + \frac{1}{} + \frac{1}{16}$
h.	$\frac{9}{16}$	$\frac{1}{} + \frac{1}{}$
i.	$\frac{11}{16}$	$\frac{1}{} + \frac{1}{} + \frac{1}{}$

j.	$\frac{13}{16}$	$\frac{1}{} + \frac{1}{} + \frac{1}{}$
k.	$\frac{3}{32}$	$\frac{1}{} + \frac{1}{}$
l.	$\frac{5}{32}$	— + —
m.	$\frac{7}{32}$	$\frac{1}{} + \frac{1}{} + \frac{1}{}$
n.	$\frac{13}{32}$	— + — + —
o.	$\frac{15}{16}$	— + — + — + —
p.	$\frac{15}{32}$	— + — + — + —
q.	$\frac{21}{32}$	— + — + —
r.	$\frac{27}{32}$	— + — + — + —

II. Circle those fractions below which can be reduced to unit fractions and cross out those which can't.

$\frac{3}{6}$ $\cancel{\frac{6}{9}}$ $\frac{6}{8}$ $\frac{4}{12}$ $\frac{6}{12}$ $\frac{8}{12}$ $\frac{2}{10}$ $\frac{4}{10}$

$\frac{12}{16}$ $\frac{6}{15}$ $\frac{25}{100}$ $\frac{22}{33}$ $\frac{7}{28}$ $\frac{9}{99}$ $\frac{13}{52}$

III. In the top row of each chart below, complete the fractions so they are already or can be reduced to unit fractions. Indicate the unit fraction in the bottom row.

	a.	b.	c.	d.	e.	f.	g.	h.	i.
1.	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{}{6}$	$\frac{}{8}$	$\frac{}{8}$	$\frac{}{8}$	$\frac{}{9}$	$\frac{}{9}$	$\frac{}{10}$
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$
2.	$\frac{}{10}$	$\frac{}{10}$	$\frac{}{12}$	$\frac{}{12}$	$\frac{}{12}$	$\frac{}{12}$	$\frac{}{12}$	$\frac{}{15}$	$\frac{}{15}$
	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$
3.	$\frac{}{15}$	$\frac{}{18}$	$\frac{}{18}$	$\frac{}{18}$	$\frac{}{18}$	$\frac{}{18}$	$\frac{}{20}$	$\frac{}{20}$	$\frac{}{20}$
	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$
4.	$\frac{}{24}$	$\frac{}{24}$	$\frac{}{24}$	$\frac{}{24}$	$\frac{}{24}$	$\frac{}{24}$	$\frac{}{24}$	$\frac{}{20}$	$\frac{}{20}$
	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$

IV. How can we obtain $\frac{7}{12}$ as a sum of unit fractions? Look in row 2 above.

$$\frac{7}{12} = \frac{6}{12} + \frac{1}{12} \quad \text{or} \quad \frac{4}{12} + \frac{3}{12} \quad \text{or} \quad \frac{4}{12} + \frac{2}{12} + \frac{1}{12}$$

Now reduce all fractions to unit fractions:

$$\frac{7}{12} = \frac{1}{2} + \frac{1}{} \quad \text{or} \quad \frac{1}{3} + \frac{1}{} \quad \text{or} \quad \frac{1}{3} + \frac{1}{} + \frac{1}{}$$

V. Find two ways to obtain $\frac{11}{12}$ as a sum of three unit fractions.

$$\frac{11}{12} = \text{—} + \text{—} + \text{—} \quad \text{or} \quad \text{—} + \text{—} + \text{—}$$

ONE DOLLAR will be our unit.

I. Suppose that we have 1 each of the regular U.S. coins less than a dollar. Indicate with unit fractions the fractional part that each coin is of the unit.

50¢	25¢	10¢	5¢	1¢
$\frac{1}{2}$	$\frac{1}{\ }$	$\frac{1}{\ }$	$\frac{1}{\ }$	$\frac{1}{\ }$

Remember that we have only 1 of each. Indicate a combination of those coins whose sum is the amount on the left.

a.	75¢	$\frac{1}{2} + \frac{1}{4}$
b.	36¢	$\frac{1}{4} + \frac{1}{\ } + \frac{1}{\ }$
c.	81¢	$\frac{\ }{2} + \underline{\ \ } + \underline{\ \ } + \underline{\ \ }$
d.	40¢	
e.	16¢	
f.	$\$\frac{3}{10}$	
g.	$\$\frac{3}{20}$	
h.	$\$\frac{31}{100}$	
i.	$\$\frac{41}{100}$	
j.	$\$\frac{7}{20}$	
k.	$\$\frac{3}{5}$	
l.	$\$\frac{9}{25}$	
m.	$\$\frac{4}{5}$	
n.	$\$\frac{14}{25}$	
o.	$\$\frac{13}{20}$	
p.	\$ .91	

II. Now suppose that we invent three new coins — a 2¢ piece, a 4¢ piece, and a 20¢ piece — all unit fractions of our dollar unit.

50¢	25¢	20¢	10¢	5¢	4¢	2¢	1¢
$\frac{1}{2}$	$\frac{1}{\ }$	$\frac{1}{\ }$	$\frac{1}{\ }$	$\frac{1}{\ }$	$\frac{1}{\ }$	$\frac{1}{\ }$	$\frac{1}{\ }$

Remember that we have only 1 of each. Indicate a combination of those coins whose sum is the amount on the left. (Some sums can be made in more than one way. Show any one you like.)

a.	35¢	$\frac{1}{4} + \frac{1}{10} = \frac{1}{5} + \frac{1}{10} + \frac{1}{20} = \frac{1}{5} + \frac{1}{10} + \frac{1}{25} + \frac{1}{100}$
b.	\$.19	
c.	$\$\frac{9}{10}$	
d.	\$.49	
e.	\$.33	
f.	$\$\frac{17}{50}$	
g.	$\$\frac{4}{5}$	
h.	$\$\frac{26}{40}$	
i.	99¢	
j.	$\$\frac{6}{8}$	
k.	$\$\frac{48}{100}$	
l.	\$.83	
m.	$\$\frac{14}{35}$	
n.	$\$\frac{51}{75}$	
o.	$\$\frac{19}{20}$	
p.	\$1.15	

You know more about the ancient Egyptian's fractions than you thought you did.

Subtracting one unit fraction from another.

I. Try a few: remember that the difference must be a unit fraction or a sum of unit fractions.

a. $\frac{1}{2} - \frac{1}{3} = \frac{1}{\quad}$

b. $\frac{1}{3} - \frac{1}{4} = \frac{1}{\quad}$

c. $\frac{1}{9} - \frac{1}{10} = \frac{1}{\quad}$

d. $\frac{1}{7} - \frac{1}{8} = \frac{1}{\quad}$

e. $\frac{1}{4} - \frac{1}{5} = \frac{1}{\quad}$

f. $\frac{1}{6} - \frac{1}{7} = \frac{1}{\quad}$

g. $\frac{1}{5} - \frac{1}{6} = \frac{1}{\quad}$

h. $\frac{1}{20} - \frac{1}{21} = \frac{1}{\quad}$

Easy! . . . after you have completed a few.

But those are all examples of a special case — the denominators are consecutive whole numbers.

II. In the green-tinted blocks below, use only unit fractions:

a. $\frac{1}{4} - \frac{1}{6} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

b. $\frac{1}{6} - \frac{1}{8} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

c. $\frac{1}{6} - \frac{1}{9} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

d. $\frac{1}{6} - \frac{1}{10} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

e. $\frac{1}{8} - \frac{1}{10} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

f. $\frac{1}{8} - \frac{1}{12} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

g. $\frac{1}{10} - \frac{1}{15} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

Do you suspect that the difference between unit fractions is always a unit fraction?

III. It is either very hard or impossible to express the difference as a unit fraction in all the starred examples that follow:

a. $\frac{1}{9} - \frac{1}{12} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

b. $\frac{1}{4} - \frac{1}{9} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$ ☆

c. $\frac{1}{10} - \frac{1}{14} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

d. $\frac{1}{9} - \frac{1}{15} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$ ☆

e. $\frac{1}{30} - \frac{1}{45} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

f. $\frac{1}{5} - \frac{1}{12} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$ ☆

g. $\frac{1}{18} - \frac{1}{24} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

h. $\frac{1}{34} - \frac{1}{51} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{1}{\quad}$

IV. Perhaps it would be easier in the three starred examples to write the difference as a sum of two different unit fractions.

$\frac{1}{4} - \frac{1}{9} = \frac{\quad}{36} - \frac{\quad}{36} = \frac{\quad}{36} = \frac{1}{\quad} + \frac{1}{36}$

$\frac{1}{9} - \frac{1}{15} = \frac{\quad}{45} - \frac{\quad}{45} = \frac{\quad}{45} = \frac{1}{30} + \frac{1}{\quad}$

$\frac{1}{5} - \frac{1}{12} = \frac{\quad}{60} - \frac{\quad}{60} = \frac{\quad}{60} = \frac{1}{\quad} + \frac{1}{\quad}$

V. None of the above differences can be expressed as a single unit fraction. The same is true of the following, but each can be expressed as the sum of two unit fractions.

a. $\frac{1}{3} - \frac{1}{7} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad} + \frac{\quad}{\quad}$

b. $\frac{1}{3} - \frac{1}{8} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad} + \frac{\quad}{\quad}$

c. $\frac{1}{4} - \frac{1}{7} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad} + \frac{\quad}{\quad}$

d. $\frac{1}{8} - \frac{1}{13} = \frac{\quad}{\quad} - \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad} + \frac{\quad}{\quad}$

Peter's Rule

"While I was doing those subtraction examples in which the denominators are consecutive whole numbers, I had an idea. I looked at the examples for a while, and I can prove that . . .

"... every unit fraction can be obtained by adding two unit fractions."

Here is Peter's argument:

"If $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ then $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$

and the same goes for every similar example. So, I have a rule that will work for every unit fraction, no matter what the denominator is. I'll let *n* hold a place for whatever number you pick as the denominator.

Since $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$ it follows

that $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$

and that's my rule! Since *n* can hold a place for any whole number (except 0), this is a proof that every unit fraction can be expressed as the sum of two unit fractions."

Do you agree that Peter has proved his rule? Has he proved that the two unit fractions in the sum must be different? When questioned about this, Peter said, "The largest unit fraction is $\frac{1}{2}$, all have denominator 2 or larger, and

if $n + 1 = n(n + 1)$ then $n = 1$."

Try Peter's rule:

$\frac{1}{8} = \frac{1}{9} + \frac{1}{\quad}$ $\qquad$ $\frac{1}{13} = \frac{1}{14} + \frac{1}{\quad}$

$\frac{1}{10} = \frac{1}{\quad} + \frac{1}{\quad}$ $\qquad$ $\frac{1}{25} = \frac{1}{\quad} + \frac{1}{\quad}$

Mabel pointed out that this rule would help express any fraction with 2 as numerator as a sum of different unit fractions. She demonstrated her idea:

$$\frac{2}{3} = \frac{1}{3} + \underbrace{\frac{1}{3}}_{} = \frac{1}{3} + \underbrace{\frac{1}{4} + \frac{1}{12}}_{}$$

by Peter's rule

$$\frac{2}{5} = \frac{1}{5} + \frac{1}{\quad} = \frac{1}{5} + \frac{1}{6} + \frac{}{\quad}$$

$$\frac{2}{11} = \frac{1}{\quad} + \frac{1}{\quad} = \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad}$$

$$\frac{2}{13} = \frac{1}{\quad} + \frac{1}{\quad} = \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad}$$

"I soon began eliminating the middle step," Mabel explained and gave more examples:

$\frac{2}{9} = \frac{1}{9} + \frac{1}{10} + \frac{1}{\quad}$ $\qquad$ $\frac{2}{15} = \frac{1}{15} + \frac{1}{\quad} + \frac{1}{\quad}$

$\frac{2}{7} = \frac{1}{7} + \frac{1}{\quad} + \frac{1}{\quad}$ $\qquad$ $\frac{2}{19} = \frac{1}{19} + \frac{1}{\quad} + \frac{1}{\quad}$

Peter's Theorem

"I can use my rule to prove that every unit fraction can be expressed as the sum of any given number of unit fractions.

"I start with a unit fraction and apply my rule. Next I apply it again to the last unit fraction of the expression, and so on as long as I wish. For example:

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6} \quad \left(\text{but } \frac{1}{6} = \frac{1}{7} + \frac{1}{42}\right)\text{, so}$$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{7} + \frac{1}{42} \left(\text{but } \frac{1}{42} = \frac{1}{43} + \frac{1}{1806}\right)\text{, so}$$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1806}$$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \frac{1}{3{,}263{,}442}$$

and I could keep on going until I had as many unit fractions as I like!"

What do you think of Peter's theorem and his proof?

A Side-trip to Ancient Rome

The Egyptians used 1 as the numerator of all fractions (except $\frac{2}{3}$) and then varied the denominators to express different amounts.

The Romans of ancient times did the reverse. They divided everything into twelfths (using 12 as the denominator) and used the numerator to indicate the number of 12ths.

I. Let's explore where this system leads. We shall begin by limiting ourselves to 12ths and to whole numbers. We shall use eleven or fewer 12ths in reporting any fractional part of the results and state our answer in simplest terms.

a. $\frac{7}{12} + \frac{8}{12} = \frac{15}{12} = 1\frac{3}{12}$

b. $\frac{3}{12} + \frac{7}{12} = \frac{}{12}$

c. $1\frac{4}{12} + 3\frac{7}{12} = 4\frac{}{12}$

d. $2\frac{6}{12} + 7\frac{6}{12} =$

e. $2\frac{8}{12} + \frac{9}{12} = 2\frac{17}{12} =$

f. $1\frac{10}{12} + 4\frac{11}{12} = \qquad =$

g. $\frac{8}{12} + \frac{10}{12} + \frac{6}{12} =$

h. $\frac{7}{12} + 1\frac{7}{12} =$

i. $\frac{9}{12} + \frac{9}{12} + \frac{9}{12} =$

j. $75 + 28\frac{6}{12} =$

Easy! So let's try subtraction:

k. $3 - \frac{7}{12} = 2\frac{}{12}$

l. $7\frac{9}{12} - 2\frac{3}{12} =$

m. $5 - \frac{4}{12} =$

n. $6 - 5\frac{1}{12} =$

o. $1\frac{8}{12} - \frac{9}{12} =$

p. $10\frac{8}{12} - 7\frac{8}{12} =$

q. $71\frac{11}{12} - 28 =$

r. $100\frac{1}{12} - \frac{2}{12} =$

s. $9\frac{3}{12} - 2\frac{6}{12} =$

t. $196 - 8\frac{9}{12} =$

Also easy! The denominators are always alike — always 12!

II. Multiplying by a whole number and dividing by a whole number can also be quite easy:

a. $\frac{5}{12} \times 3 = \frac{}{12} =$

b. $6\frac{8}{12} \div 2 =$

c. $2\frac{2}{12} \times 7 = \qquad =$

d. $1\frac{2}{12} \div 2 =$

e. $3\frac{5}{12} \times 5 =$

f. $1\frac{3}{12} \div 3 =$

g. $8\frac{7}{12} \times 6 =$

h. $1\frac{3}{12} \div 5 =$

i. $4 \times 9\frac{5}{12} =$

j. $2\frac{4}{12} \div 7 =$

k. $8 \times 12\frac{6}{12} =$

l. $43\frac{2}{12} \div 7 =$

Look back through these examples. Think of each as a question involving feet and inches.

a. How much is 5 inches $\times$ 3?

b. How much is (6 feet and 8 inches) $\div$ 2?

c. How much is (2 feet and 2 inches) $\times$ 7?

Etc.

Any difficulty you may have had soon disappears.

III. Take half of it . . . or $\frac{6}{12}$ of it . . . or $\times\ \frac{6}{12}$.

a. $10 \times \frac{6}{12} =$

b. $1 \times \frac{6}{12} =$

c. $\frac{8}{12} \times \frac{6}{12} =$

d. $1\frac{6}{12} \times \frac{6}{12} =$

IV. Take a third of it . . . or $\frac{4}{12}$ of it . . . or $\times\ \frac{4}{12}$.

a. $1 \times \frac{4}{12} =$

b. $\frac{9}{12} \times \frac{4}{12} =$

c. $2\frac{3}{12} \times \frac{4}{12} =$

d. $\frac{3}{12} \times \frac{4}{12} =$

V. Take two-thirds of it . . . or $\frac{8}{12}$ of it . . . or $\times\ \frac{8}{12}$.

a. $3 \times \frac{8}{12} =$

b. $\frac{9}{12} \times \frac{8}{12} =$

c. $1\frac{3}{12} \times \frac{8}{12} =$

d. $2\frac{6}{12} \times \frac{8}{12} =$

e. $6\frac{6}{12} \times \frac{8}{12} =$

f. $4\frac{9}{12} \times \frac{8}{12} =$

A little strange — but still easy.

VI. How many inches in 2 feet, 1 foot, half a foot, a third of a foot? How many 12ths of a foot are there?

a. $1 \div \frac{1}{12} =$ b. $2 \div \frac{1}{12} =$

c. $\frac{6}{12} \div \frac{1}{12} =$ d. $1\frac{6}{12} \div \frac{1}{12} =$

e. $\frac{7}{12} \div \frac{1}{12} =$ f. $3\frac{5}{12} \div \frac{1}{2} =$

VII. How many 2-inch lengths in a foot? How many $\frac{2}{12}$-of-a-foot lengths are there in 10 inches?

a. $1 \div \frac{2}{12} =$ b. $2 \div \frac{2}{12} =$

c. $\frac{6}{12} \div \frac{2}{12} =$ d. $1\frac{8}{12} \div \frac{2}{12} =$

VIII. How many 3-inch lengths in 2 feet, in 1 foot?

a. $2 \div \frac{3}{12} =$ b. $1 \div \frac{3}{12} =$

c. $\frac{6}{12} \div \frac{3}{12} =$ d. $1\frac{9}{12} \div \frac{3}{12} =$

IX. How many 6-inch strips in 1 foot, in 6 inches, in 3 inches, in 9 inches?

a. $1 \div \frac{6}{12} =$ b. $\frac{6}{12} \div \frac{6}{12} =$

c. $\frac{3}{12} \div \frac{6}{12} =$ d. $\frac{9}{12} \div \frac{6}{12} =$

e. $\frac{2}{12} \div \frac{6}{12} =$ f. $\frac{1}{12} \div \frac{6}{12} =$

X. How many 1 foot 6 inch lengths are there in 3 feet, in 6 feet, in 9 inches, in 6 inches?

a. $3 \div 1\frac{6}{12} =$ b. $6 \div 1\frac{6}{12} =$

c. $\frac{9}{12} \div 1\frac{6}{12} =$ d. $\frac{6}{12} \div 1\frac{6}{12} =$

A little harder? A little, but not much.

Of course, we've kept the examples on the simple side so you could see more clearly how the system works.

What's inconvenient about this system? Aren't you happy you don't have to say "six-twelfths of an hour"?

But how would you refer to a minute? Five minutes is $\frac{1}{12}$ of an hour.

Suppose that you wished to measure a length which was more than $\frac{6}{12}$ of a foot but less than $\frac{7}{12}$ of a foot. It would seem logical to divide each inch into 12 parts so that you could say that the length was $\frac{6}{12}$ plus, say, $\frac{3}{12}$ of $\frac{1}{12}$, or plus $\frac{3}{144}$.

$$\frac{6}{12} + \frac{3}{144} = \frac{72}{144} + \frac{3}{144} = \frac{75}{144}$$

Perhaps you would like to think about such arithmetic and explore the problems you would encounter. You might, in a reference book, look up the "duodecimal system," a system based on twelve and powers of twelve as our decimal system is based on ten and powers of ten.

But we wish to complete this side-trip at this point.

Before we leave it completely, we would like to mention that we do have a system of fractions that, like the system of early Romans, has just one denominator. That denominator is 100 and we use the word *percent* or the symbol % to indicate that we are using that system.

In this system we change everything to hundredths and then use % instead of writing a denominator. You are probably familiar with it, but for those who aren't, here are some examples:

$\frac{1}{2} = \frac{\quad}{100} = 50\%$ $\frac{1}{4} = \frac{\quad}{100} = \quad\%$

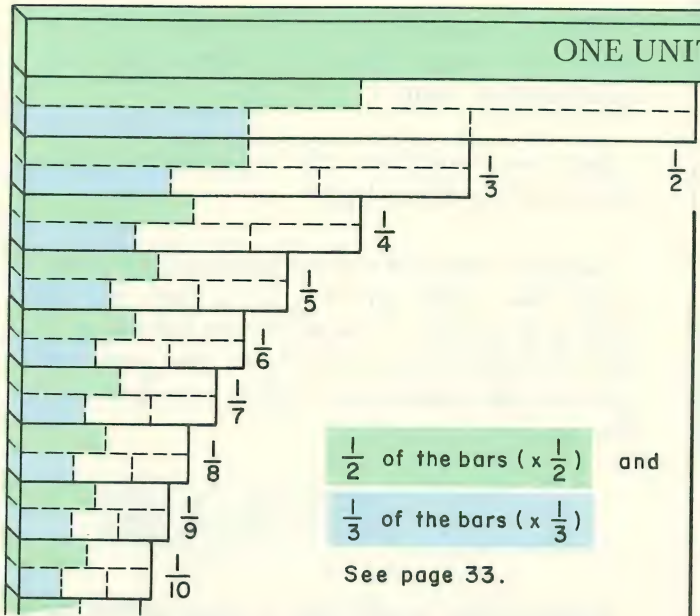
$\frac{3}{4} = \frac{\quad}{100} = \quad\%$ $\frac{1}{5} = \frac{\quad}{100} = \quad\%$

$\frac{2}{5} = \frac{\quad}{100} = \quad\%$ $\frac{7}{10} = \frac{\quad}{100} = \quad\%$

$1\frac{1}{2} = \frac{\quad}{100} = \quad\%$ $2 = \frac{\quad}{100} = \quad\%$

$1\frac{3}{10} = \frac{\quad}{100} = \quad\%$ $\frac{3}{25} = \frac{\quad}{100} = \quad\%$

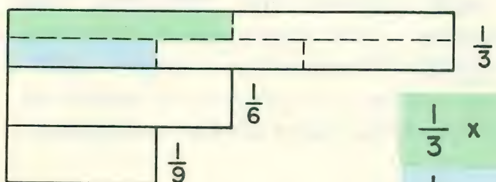
Back from our side trip to ancient Rome, let's take a brief look at multiplication and division involving unit fractions and whole numbers.



$\frac{1}{2}$ of the bars (x $\frac{1}{2}$) and

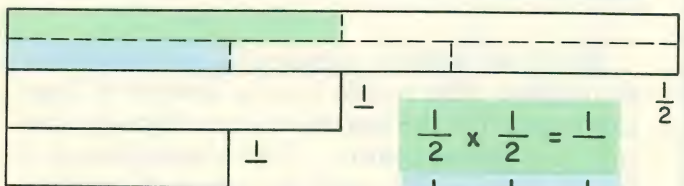
$\frac{1}{3}$ of the bars (x $\frac{1}{3}$)

See page 33.



$\frac{1}{3} \times \frac{1}{2} = \frac{1}{\_\_}$

$\frac{1}{3} \times \frac{1}{3} = \frac{1}{\_\_}$

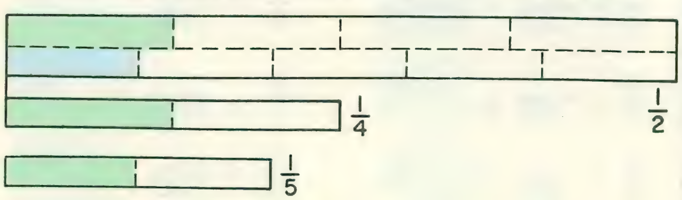


$\frac{1}{2} \times \frac{1}{2} = \frac{1}{\_\_}$

$\frac{1}{2} \times \frac{1}{3} = \frac{1}{\_\_}$

$\frac{1}{4} \times \frac{1}{2} = \frac{1}{\_\_}$

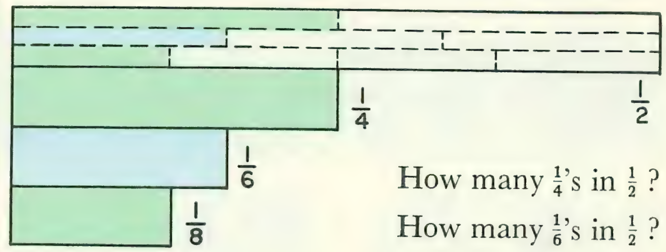
$\frac{1}{4} \times \frac{1}{3} = \frac{1}{\_\_}$



$\frac{1}{2} \times \frac{1}{4} = \frac{1}{\_\_}$ $\qquad$ $\frac{1}{4} \times \frac{1}{2} = \frac{1}{\_\_}$

$\frac{1}{2} \times \frac{1}{5} = \frac{1}{\_\_}$ $\qquad$ $\frac{1}{5} \times \frac{1}{2} = \frac{1}{\_\_}$

There is certainly nothing new yet in multiplying a pair of unit fractions. All multiplication should be as simple!



How many $\frac{1}{4}$'s in $\frac{1}{2}$?

How many $\frac{1}{6}$'s in $\frac{1}{2}$?

How many $\frac{1}{8}$'s in $\frac{1}{2}$?

$\frac{1}{2} \div \frac{1}{4} =$ $\qquad$ $\frac{1}{2} \div \frac{1}{6} =$ $\qquad$ $\frac{1}{2} \div \frac{1}{8} =$

5	1	$\frac{1}{\_\_}$	$\frac{1}{\_\_}$	$\frac{1}{\_\_}$	$\frac{1}{\_\_}$	$\frac{1}{\_\_}$

How many quarters in a half?
How many halves in a quarter?

$\frac{1}{2} \div \frac{1}{4} =$ $\qquad$ $\frac{1}{4} \div \frac{1}{2} =$

How many dimes in a half?
How many halves in a dime?

$\frac{1}{2} \div \frac{1}{10} =$ $\qquad$ $\frac{1}{10} \div \frac{1}{2} =$

How many pennies in a quarter?
How many quarters in a penny?

$\frac{1}{4} \div \frac{1}{100} =$ $\qquad$ $\frac{1}{100} \div \frac{1}{4} =$

How many quarters in five dollars?
How many five dollars in a quarter?

$5 \div \frac{1}{4} =$ $\qquad$ $\frac{1}{4} \div 5 =$

How many dimes in a dollar?
How many dollars in a dime?

$1 \div \frac{1}{10} =$ $\qquad$ $\frac{1}{10} \div 1 =$

Again . . . nothing new, so let's use some sums of unit fractions and whole numbers.

$\begin{array}{r} 12 + \frac{1}{2} + \frac{1}{5} \\ \times \frac{1}{3} \\ \hline 4 + \frac{1}{6} + \frac{1}{15} \end{array}$	$\begin{array}{r} 35 + \frac{1}{6} + \frac{1}{9} + \frac{1}{13} \\ \times \frac{1}{7} \\ \hline 5 + \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad} \end{array}$
$\begin{array}{r} 6 + \frac{1}{8} + \frac{1}{15} \\ \times \frac{1}{5} \\ \hline 1 + \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad} \end{array}$	$\begin{array}{r} 20 + \frac{1}{18} + \frac{1}{64} \\ \times \frac{1}{10} \\ \hline \quad + \frac{1}{\quad} + \frac{1}{\quad} \end{array}$

No trouble at all, if you thought of 6 as 5 + 1 and noticed:

$$\begin{array}{r} 5 + 1 \\ \times \frac{1}{5} \\ \hline 1 + \frac{1}{5} \end{array}$$

In fact, it's usually easier to multiply by a unit fraction than by a whole number. Try the next two on your own and without many clues.

$\begin{array}{r} 3 + \frac{1}{4} + \frac{1}{6} + \frac{1}{20} \\ \times 2 \\ \hline 6 + \frac{2}{4} + \text{—} + \text{—} \\ \hline 6 + \frac{1}{2} \end{array}$	$\begin{array}{r} 3 + \frac{1}{4} + \frac{1}{6} + \frac{1}{20} \\ \times 3 \\ \hline 9 + \frac{3}{4} + \text{—} + \text{—} \\ \hline ? \end{array}$

The first example is too simple. The second presents real trouble. Only one of the "partial products" can be reduced to a unit fraction. The other two need more work:

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{\quad} \qquad \frac{3}{20} = \frac{1}{10} + \frac{1}{\quad}$$

So, we write out:

$$9 + \frac{1}{2} + \frac{1}{4} + \frac{1}{2} + \frac{1}{10} + \frac{1}{20}$$

But we still are in trouble. The unit fractions are not all different, and their sum is greater than 1. There's an easy way out because the sum of the two fractions with denominator 2 is a whole number. So we can write the product according to all the rules:

$$\quad + \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad}$$

Try two examples completely on your own. Use the space not tinted green to keep track of your work.

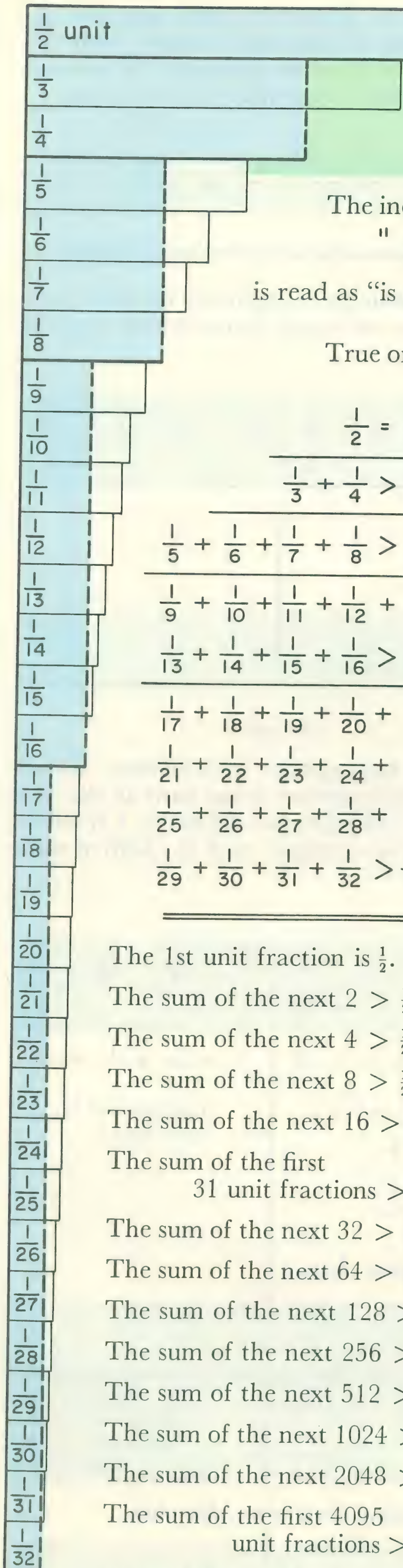
$\begin{array}{r} 7 + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} \\ \times 2 \\ \hline \\ \hline \\ \end{array}$	$\begin{array}{r} \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \\ \times 3 \\ \hline \\ \hline \\ \end{array}$

Division

It's quite the opposite with division. Dividing by a whole number is not hard at all. The difficulty arises when we divide by a fraction. Let's try a few examples with the help of some clues.

$\begin{array}{r} 3 + \frac{1}{2} + \frac{1}{4} + \frac{1}{3} \\ \div \frac{1}{2} \\ \hline 6 + 1 + \frac{1}{\quad} + ? \end{array}$ Think of $\frac{1}{3}$ as $\frac{1}{6} + \frac{1}{6}$. $\begin{array}{r} \frac{1}{6} + \frac{1}{6} \\ \div \frac{1}{2} \\ \hline \frac{1}{3} + \quad \end{array}$ $6 + 1 + \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad}$	$\begin{array}{r} \frac{1}{8} + \frac{1}{12} + \frac{1}{16} + \frac{1}{20} \\ \div \frac{1}{4} \\ \hline \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad} \end{array}$ But that sum is more than 1.
$\begin{array}{r} \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \\ \div \frac{1}{3} \\ \hline \end{array}$	$\begin{array}{r} 7 + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} \\ \div \frac{1}{2} \\ \hline \end{array}$

Before working, look above for clues.



What is the sum of all unit fractions?

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\ldots \text{ etc.}$$

The inequality sign:

" $>$ "

is read as "is greater than."

True or False

$\frac{1}{2}=\frac{1}{2}$	T	F
$\frac{1}{3}+\frac{1}{4}>\frac{1}{2}$	T	F
$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{2}$	T	F
$\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}>\frac{1}{2}$	T	F
$\frac{1}{17}+\frac{1}{18}+\frac{1}{19}+\frac{1}{20}+\frac{1}{21}+\frac{1}{22}+\frac{1}{23}+\frac{1}{24}+\frac{1}{25}+\frac{1}{26}+\frac{1}{27}+\frac{1}{28}+\frac{1}{29}+\frac{1}{30}+\frac{1}{31}+\frac{1}{32}>\frac{1}{2}$	T	F

The 1st unit fraction is $\frac{1}{2}$.	T	F
The sum of the next 2 $>\frac{1}{2}$.	T	F
The sum of the next 4 $>\frac{1}{2}$.	T	F
The sum of the next 8 $>\frac{1}{2}$.	T	F
The sum of the next 16 $>\frac{1}{2}$.	T	F
The sum of the first 31 unit fractions $>2\frac{1}{2}$.	T	F
The sum of the next 32 $>\frac{1}{2}$.	T	F
The sum of the next 64 $>\frac{1}{2}$.	T	F
The sum of the next 128 $>\frac{1}{2}$.	T	F
The sum of the next 256 $>\frac{1}{2}$.	T	F
The sum of the next 512 $>\frac{1}{2}$.	T	F
The sum of the next 1024 $>\frac{1}{2}$.	T	F
The sum of the next 2048 $>\frac{1}{2}$.	T	F
The sum of the first 4095 unit fractions >6.	T	F
We could keep this up as long as we wish.	T	F

The sum of the first three unit fractions is more than 1 unit.

Is the sum of all unit fractions less than $\frac{1}{4}$ and greater than $\frac{1}{33}$ more or less than another whole unit?

Guess. Don't try to carry out the addition because you will quickly encounter very large common denominators. Start the computation, but stop as soon as you see the difficulty.

Actually, you can answer the question without any difficult computations at all.

Study the questions on the left using the diagram to help you.

Are you convinced that the sum of the first 31 unit fractions is more than $2\frac{1}{2}$ units?

If the final statement in the left-hand column is true, what do you have to say about the question we started with:

What is the sum of all unit fractions?

The sum of the first (2^1-1) unit fractions $=\frac{1}{2}$.

The sum of the first (2^2-1) unit fractions $>$ ___1___.

The sum of the first (2^3-1) unit fractions $>$ ____.

The sum of the first (2^4-1) unit fractions $>$ ____.

The sum of the first (2^5-1) unit fractions $>$ ____.

The sum of the first $(2^{50}-1)$ unit fractions $>$ ______.

The sum of the first $(2^{1000}-1)$ unit fractions $>$ ______.

For each $n>0$,
the sum of the first (2^n-1) unit fractions $>\frac{n}{2}$.

Can you say that the sum of all unit fractions is larger than any number you can think of?

Do you feel that we have proved that it must be true?

As you were exploring this problem out of the past, did you realize that by the fourth or fifth page of this chapter, you knew more about unit fractions than most of the well educated people in the world?

We do not say this primarily as a compliment to you. Rather, we wish to illustrate something about the subject of mathematics:

Select a problem and work at it. Before you know it, you are learning something that very few people know about.

But don't think that you have learned all there is to know about unit fractions! You've only touched the subject lightly.

As the authors were working on this chapter, many problems arose that we did not raise with you. Some of them might be interesting for you to work at. We shall mention one.

Which *common* fractions (smaller than 1) can be expressed as the sum of two unit fractions?

No fraction larger than $\frac{5}{6}$ can be expressed as the sum of two unit fractions. Why? Because the sum of the two largest is $\frac{5}{6}$.

But $\frac{4}{5}$, which is smaller than $\frac{5}{6}$, cannot be expressed as the sum of two unit fractions. We experiment:

$$\frac{4}{5} = \frac{1}{2} + \frac{3}{10} \qquad \frac{4}{5} = \frac{1}{3} + \frac{7}{15}$$

If we try $\frac{1}{2}$ as the first fraction, the second must be $\frac{3}{10}$. But this isn't a unit fraction. If we try $\frac{1}{3}$ as the first, the second must be $\frac{7}{15}$ — not a unit fraction. Besides, it's larger than $\frac{1}{3}$.

So, we have nothing more to do but report that more than two unit fractions will be required.

Here are some other fractions, less than $\frac{5}{6}$, that call for more than two unit fractions:

$$\frac{3}{7} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28} \qquad \frac{7}{9} = \frac{1}{2} + \frac{1}{6} + \frac{1}{9}$$

$$\frac{5}{7} = \frac{1}{2} + \frac{1}{7} + \frac{1}{14} \qquad \frac{5}{11} = \frac{1}{3} + \frac{1}{11} + \frac{1}{33}$$

(We show one of *several* possible combinations using three unit fractions.)

What is unusual here? Are all numerators and denominators prime numbers? No, 9 is not a prime.

Peter's rule (page 37) showed that every unit fraction can be expressed as the sum of two unit fractions.

We can also show that every fraction with 2 as a numerator can be so expressed. (Of course, the denominator would be odd — otherwise we would have reduced the fraction.) The method is this:

$$\frac{2}{7} = \frac{1}{4} + \frac{1}{28} \qquad \frac{2}{9} = \frac{1}{5} + \frac{1}{45} \qquad \frac{2}{11} = \frac{1}{6} + \frac{1}{66}$$

$2 \times 4 = 7 + 1$ $\qquad$ $2 \times 5 = 9 + 1$ $\qquad$ $2 \times 6 = 11 + 1$

$7 \times 4 = 28$ $\qquad$ $9 \times 5 = 45$ $\qquad$ $11 \times 6 = 66$

You try some:

$$\frac{2}{13} = \frac{1}{} + \frac{1}{} \qquad \frac{2}{19} = \frac{1}{} + \frac{1}{}$$

Let's try other even numerators:

$$\frac{4}{11} = \frac{1}{} + \frac{1}{} \qquad \frac{4}{13} = \frac{1}{} + \frac{1}{}$$

The first is encouraging — the second is a counterexample.

Perhaps you would like to explore this problem further. It won't be easy.

Chapter SEVEN . . . SYSTEMS FOR LOCATING THINGS IN SPACE

If someone knows your address, he can drop a properly addressed card into almost any mail box in the world and it will eventually reach you — provided that it has enough postage on it.

This is possible because of an address system that may require (1) the country you live in, (2) the state, (3) the city, (4) the street, (5) the house number on that street and, perhaps, (6) an apartment number.

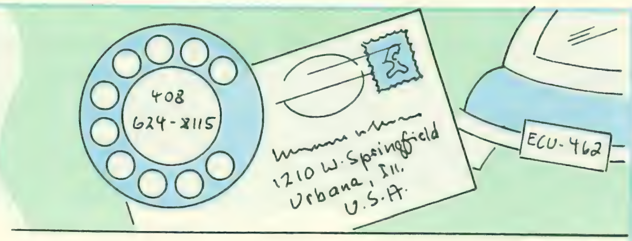
Some people can be located by their telephone number: a 3-digit area code, a 3-digit exchange, and a 4-digit local number — such as 408-624-8115.

The owner of a car can be located through his license number:

Calif. ECU-462

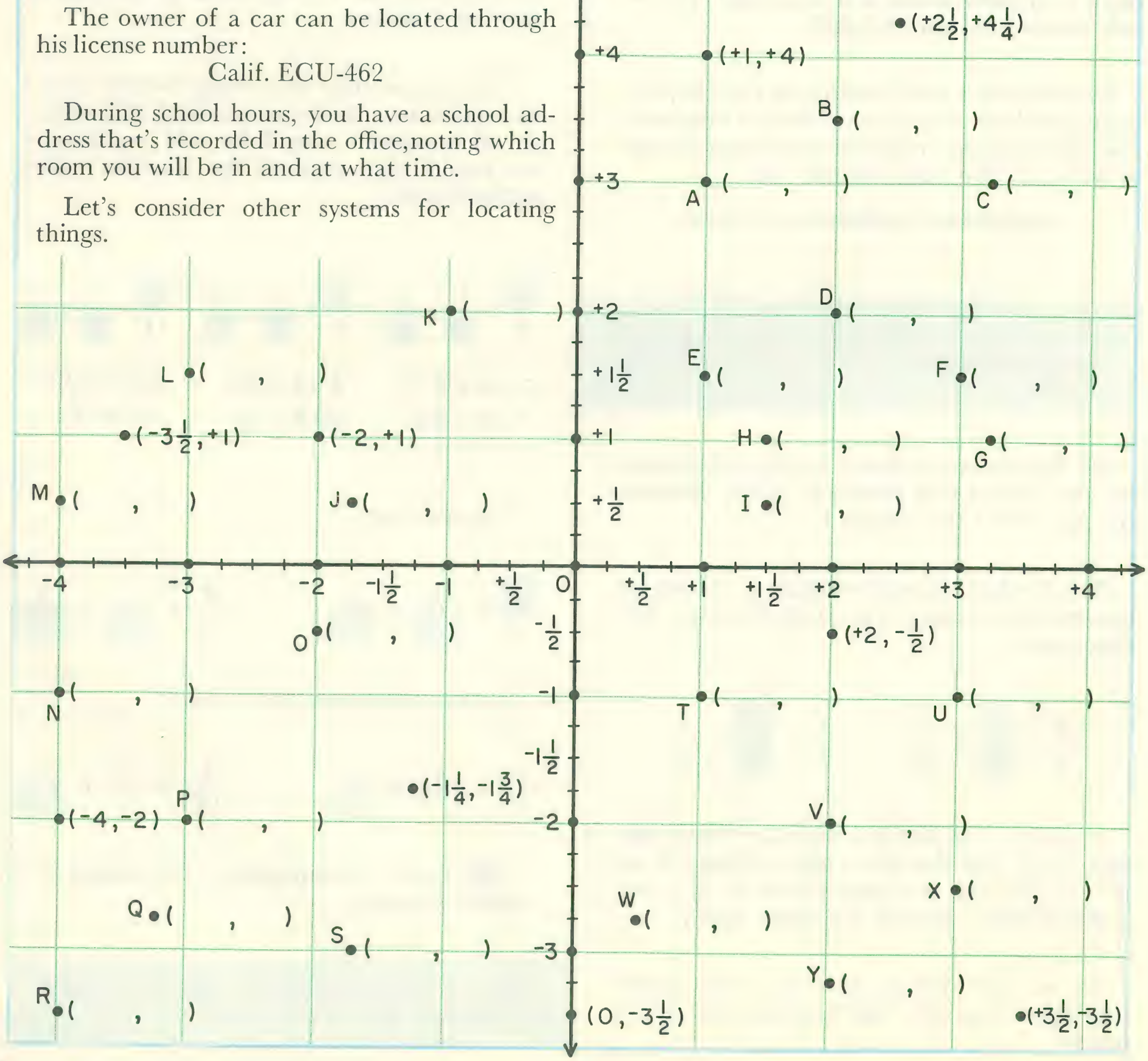
During school hours, you have a school address that's recorded in the office, noting which room you will be in and at what time.

Let's consider other systems for locating things.



On this page we have used a system that gives an address for every point inside the blue borders. Our addressing system requires only two "signed" numbers.

Study those points we have labeled with their addresses. When you think that you have figured out our system, use it to find the addresses of the other points (the ones which have letters next to them). Write the addresses next to the points.



An Alternate System of Addresses

(+90°, 6.4") or (-270°, 6.4")

(+120°, 7") or (-240°, 7")

A

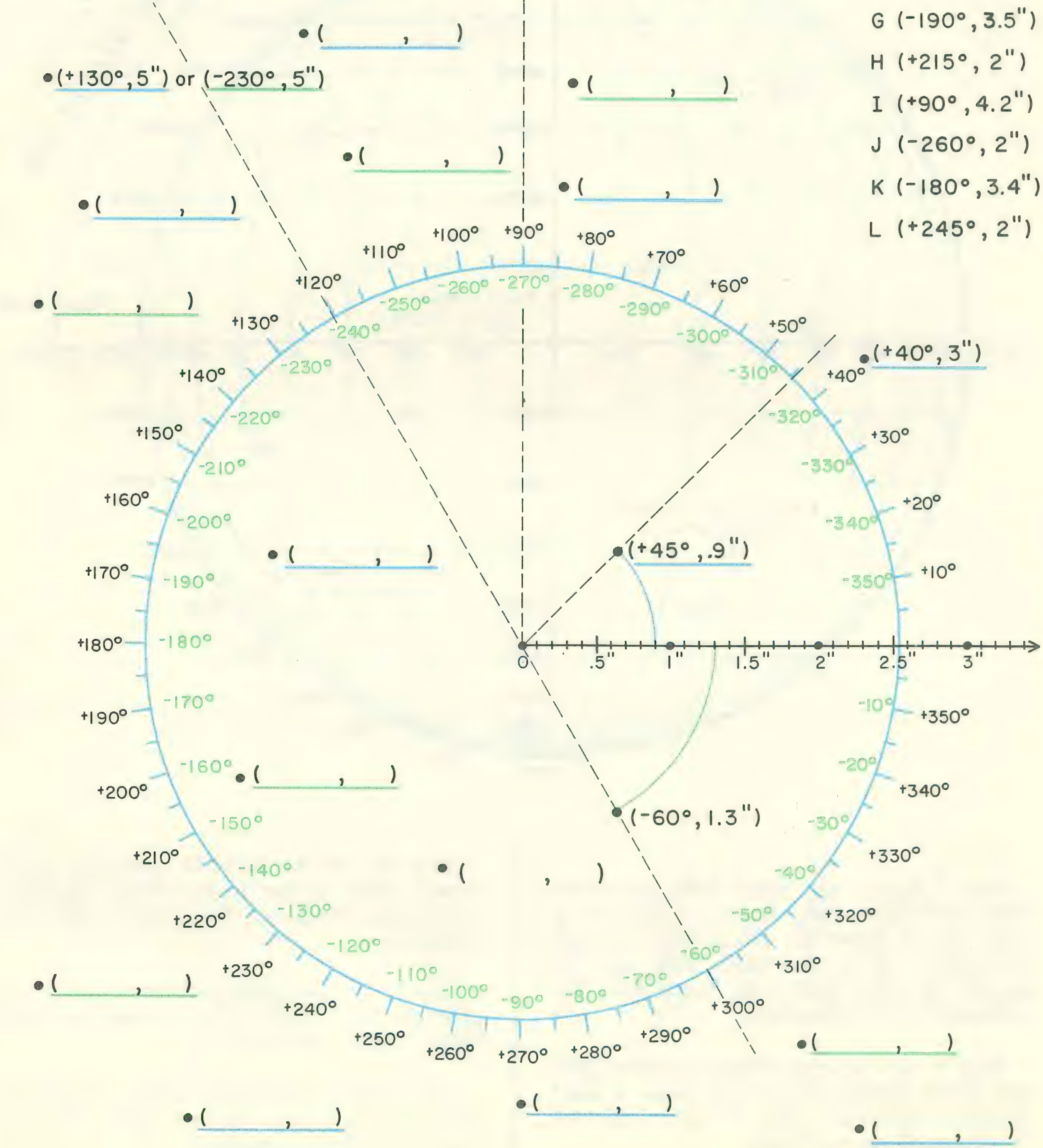
Study the points labeled with addresses. From these clues, can you figure out the system well enough to find addresses for all the points?

Each point has two addresses, but no address fits more than one point.

I. Write an address next to each point (dot), and be sure to follow the color scheme in choosing the address.

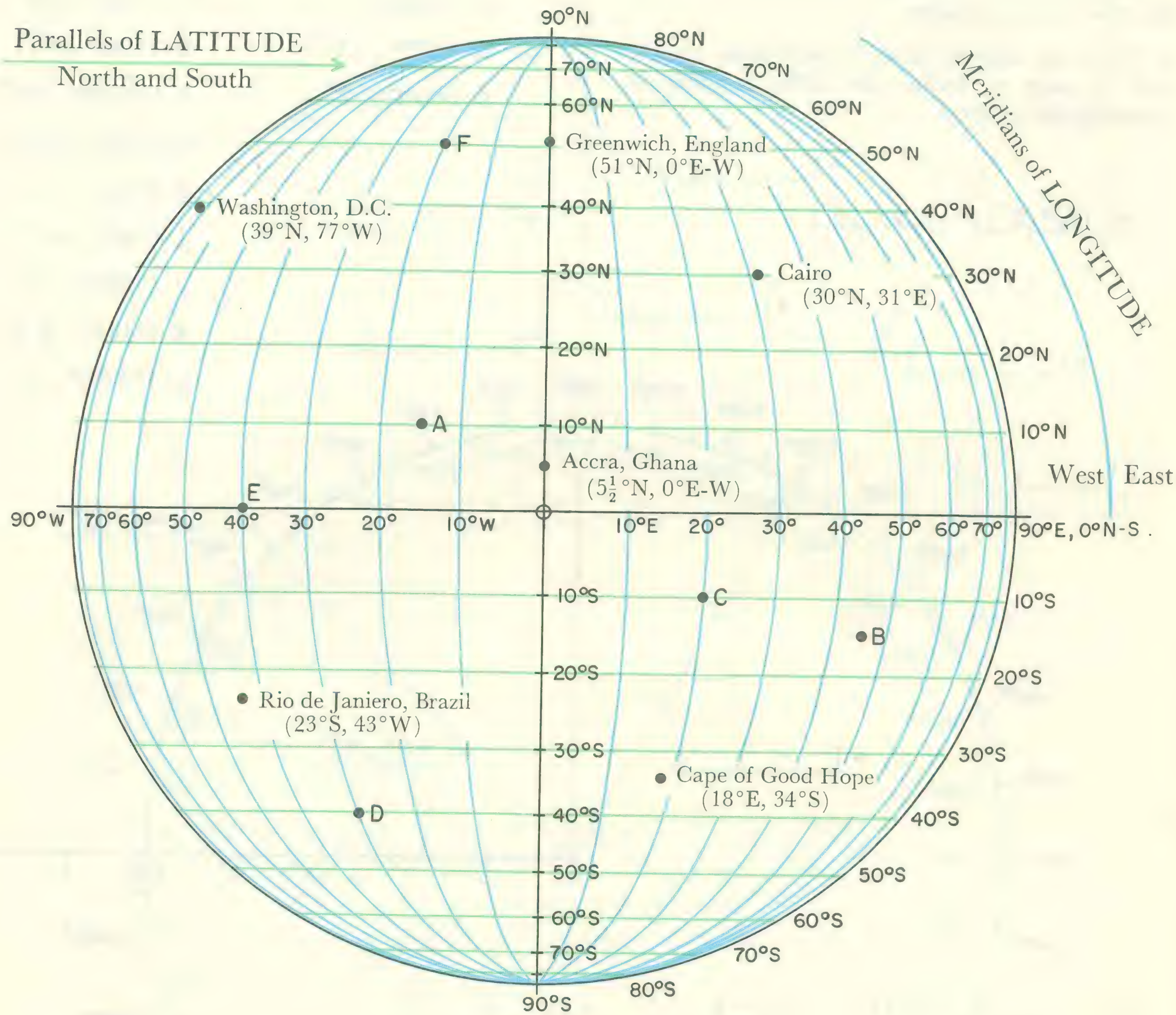
II. Please mark with dots the points that have the following addresses and label them with the appropriate letter.

A (-300°, 7")
B (+75°, 1.5")
C (+180°, 2")
D (-135°, 4.8")
E (-90°, .4")
F (+325°, 4.1")
G (-190°, 3.5")
H (+215°, 2")
I (+90°, 4.2")
J (-260°, 2")
K (-180°, 3.4")
L (+245°, 2")



Locating Points on a Sphere such as the Earth.

There are, of course, many systems we might choose to give every point a geographical address. However, the one in most common use since Ptolemy in the second century A.D. is based on imagining two sequences of circles.



Step I. Locate the North Pole, the South Pole, and Greenwich, England. Imagine a circle drawn from the North Pole through Greenwich, on to the South Pole, and on around the world back to the North Pole. We call this *the Prime Meridian.*

Step II. On the semicircle from pole to pole that passes through Greenwich, locate a point halfway between the poles. We have indicated it with an O on our sketch.

Step III. At this point O, imagine a circle drawn perpendicular to the Prime Meridian and around the globe. This circle is called *the Equator.*

Step IV. Draw imaginary circles around the globe that are parallel to the Equator. We call them *parallels of latitude.*

Step V. Draw imaginary circles around the globe that pass through both poles. We call them *meridians of longitude.*

Step VI. Divide the Equator and the Prime Meridian each into 360 parts of the same length. We call each part *a degree.*

Step VII. Number the parallels of latitude by the number of degrees from the Equator. The North Pole is 90° north, and the South Pole is 90° south.

Step VIII. Number the meridians of longitude from the Prime Meridian, to the west and to the east. Half-way around the Equator from the Prime Meridian is 180° east and 180° west.

If we use 25,000 miles as the circumference of the earth then a degree of latitude must be about _____ miles (to the nearest mile).

Give the geographical address of the points labeled with letters on the sketch on the previous page.

A. (10°N, 15°W)	B. (15°S, 45°E)
C. (_______)	D. (_______)
E. (_______)	F. (_______)

In the sketch on the previous page, locate and label the following points:

G. (10°S, 50°W)	H. (15°N, 20°E)
I. (20°N, 25°W)	J. (50°S, 30°E)
K. (5°S, 5°E)	L. ($27\frac{1}{2}$°N, 60°W)

A bear started at a certain point, walked 5 miles south, then 5 miles east, and then 5 miles north. He had walked 15 miles and had returned to his starting point.

What was the color of the bear? _________

This is obviously a kind of trick question, but think about it for a few minutes.

A Refinement of this System

The latitude-longitude system would be sufficient if the world were perfectly spherical and perfectly smooth. It is neither.

To help locate points better, we can add another bit of information to their addresses.

We shall locate points in terms of their distance (usually in feet) above or below sea level — midway between low tide and high tide, called *Mean Sea Level* (MSL). We shall write 100 ft. above sea level as "$^{+}$100 ft." and 100 feet below sea level as "$^{-}$100 feet."

The lowest point in the United States has this address:

(35°N, 117°W, $^{-}$282 ft.)

It is ______________________________

The highest point in the United States has this address:

(73°N, 150°W, $^{+}$20,320 ft.)

It is ______________________________

There is a large city in the United States that is below sea level. It's address is:

(29°N, 90°W, $^{-}$5 ft.)

It is ______________________________

The deepest known point on the floor of the sea has this address:

(12°N, 140°E, $^{-}$36,198 ft.)

It is in the Mariana Trench which is about 1,400 miles east of the __________ Islands.

The labeling center of this addressing system would have the address:

(0° N-S, 0° E-W, _____ ft.)

and is located about _____ miles south of Accra, Ghana.

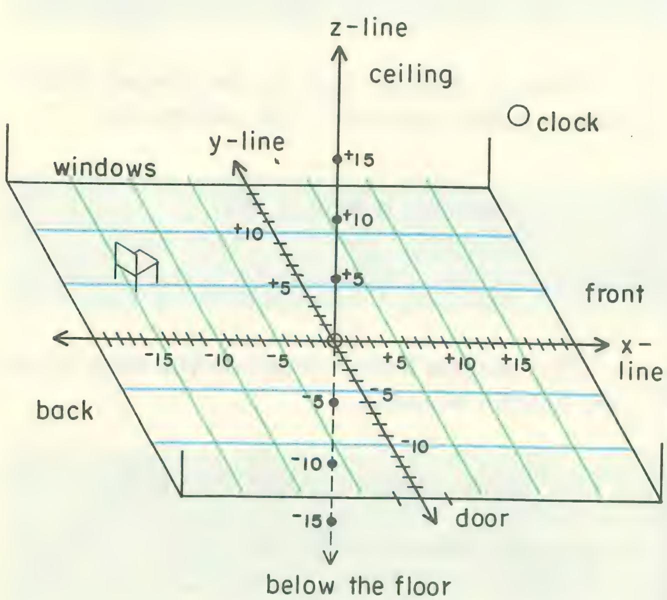
Locating points in a room . . . and beyond.

Let's consider a system that will enable us to find an address for every point in a classroom. We shall construct the system so it can be extended to every point in the building. This system will give every point its very own address.

Consider a room that is 30 ft. by 40 ft. It has windows on the left and a door on the right as you face the front. We shall start our job by locating the center of the floor.

Next we draw 3 imaginary lines and mark them off in feet. One is through the center of the floor and parallel to the window wall — our "*x*-line." The second is through the center of the floor and perpendicular to the *x*-line — our "*y*-line." The third is through the center and perpendicular to both the *x*-line and *y*-line. This is our "*z*-line." The *z*-line, in our imagination, goes up through the ceiling and down through the floor.

Here's a sketch:



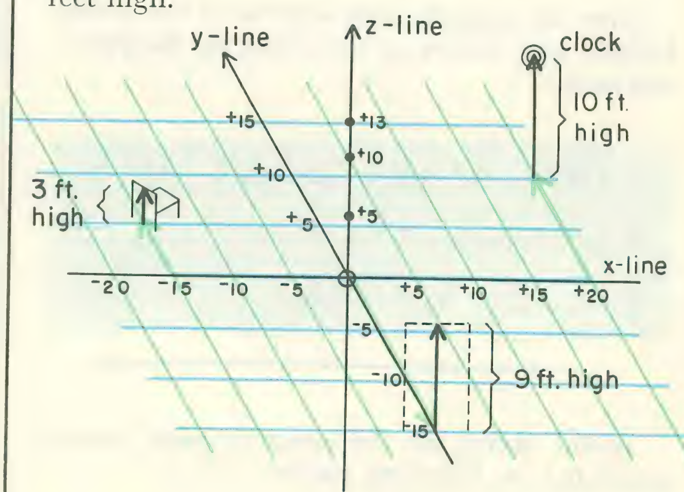
Our addressing system, similar to that on page 44, uses the positive number sign to indicate direction from our center toward the front (*x*-line), toward the windows (*y*-line), and toward the ceiling (*z*-line). The negative number sign indicates direction toward the back (*x*-line), toward the wall with the door (*y*-line), and below the floor (*z*-line).

Our addresses will all have 3 parts. The first tells us how far to move toward the front or toward the back of the room.

The second tells us how far to move and in what direction to move on a line parallel to the *y*-line.

The third tells us how far to move and whether up or down on a line parallel to the *z*-line.

Let's try our system on three points — the center of the clock face, 10 feet high; the midpoint of the top of the door, 9 feet high; and the midpoint of the top of the chair back, 3 feet high.



(distances below are in feet)

	x-line	*y*-line	*z*-line
Center of clock	+20	+10	+10
Top of door	0		
Top of chair back			
Center of floor			

What can you say about the point whose address is ($^{+}7$, $^{+}20$, $^{-}8$)?

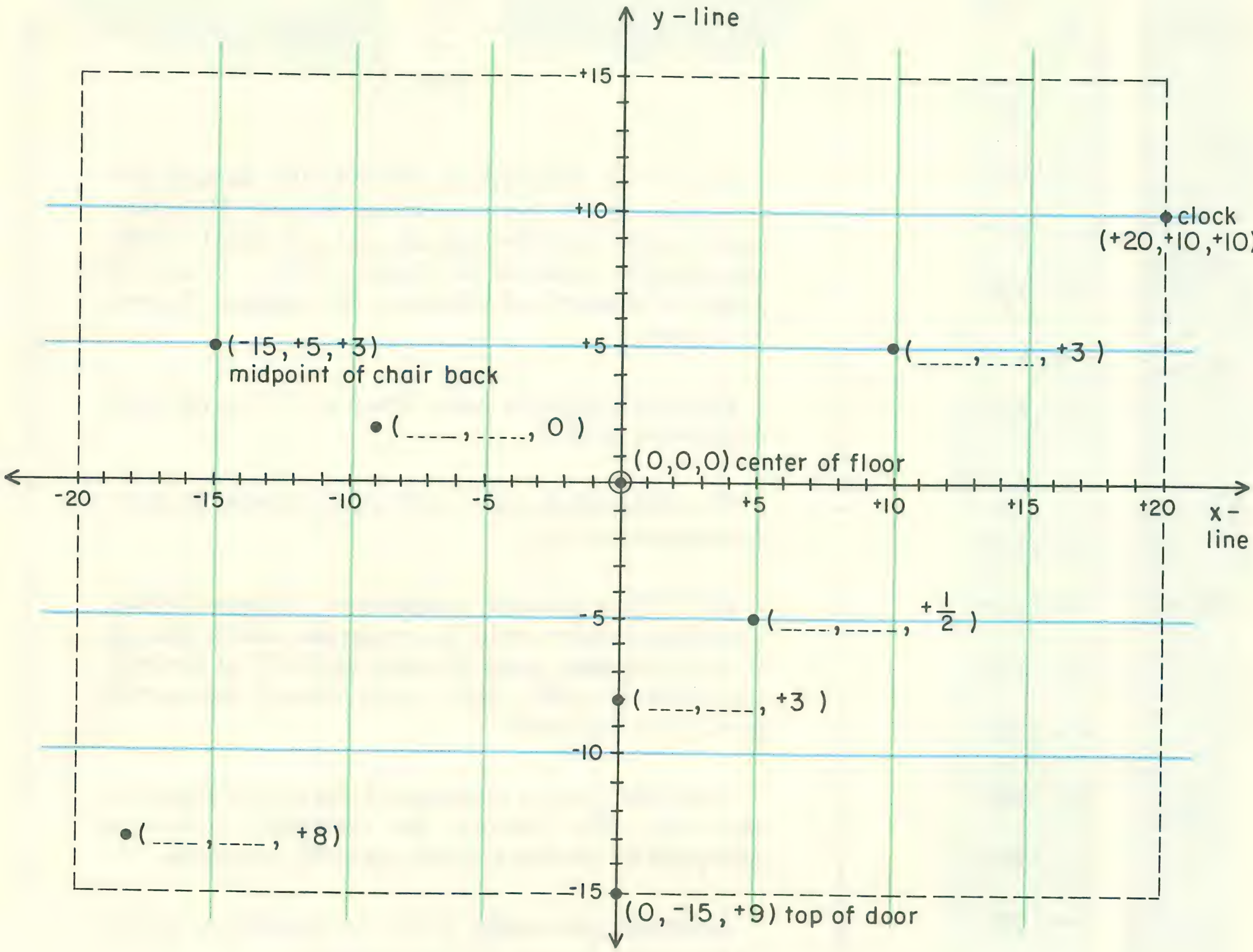
____

If the ceiling is 13 ft. high, give the address of the point on the ceiling above O on the floor.

(___ , ___ , ___)

It's hard to make a sketch in two dimensions (length and width of a piece of paper) of a room that has three dimensions — length, width, and height.

It's easy, however, to make a floor plan and give addresses to all points on it (page 44). Then the complete address can indicate the height from the floor.



Please complete the address-labels that now indicate only the number of feet above the floor.

Can you locate the following points? When you do, label them with the letters given below.

A (+5 , +10, +7)	B (+15, -10, +7½)
C (-10, +10, 0)	D (-10, -5 , +3)
E (-5, -10, +11)	F (+15, +9, +9)
G (0, -13, 0)	H (+14, -14, +8)
I (-19, 0, +8)	J (-15, +12, 0)
K (+2, +4, 0)	L (+18, 0, 0)

Assuming that the unit used above is one foot, use any method you like to find the distance to the nearest foot between the pairs of points whose address-labels are given below (be careful with the last three examples):

(+15, 0, 0) and (-8, 0, 0) 23 ft.

(-3, +5, +8) and (-3, +9, +8) ____ ft.

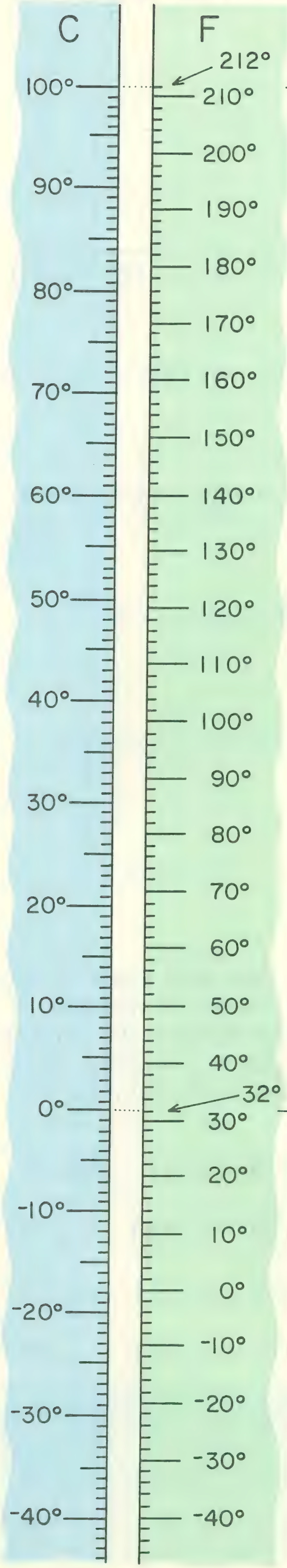
(-7, -6, +5) and (-7, -6, 0) ____ ft.

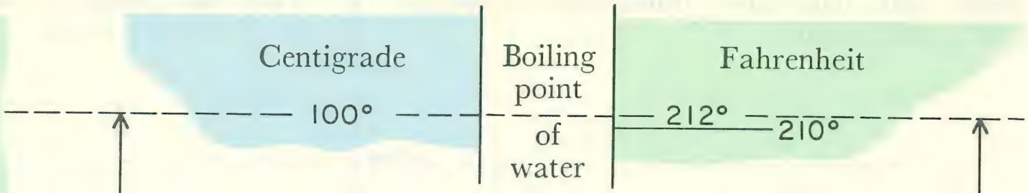
(+3, 0, +7) and (0, +4, +7) ____ ft.

(-5, 0, +12) and (0, 0, 0) ____ ft.

(-15, -10, +8) and (-9, -10, 0) ____ ft.

(-3, 0, +12) and (0, -4, 0) ____ ft.





100° C

180° F

Gabriel D. Fahrenheit (1687-1736) located the zero point on his thermometer as follows. He mixed equal weights of snow and salt and said that 0° is the temperature at which the mixture froze. From this point he marked off distances to measure degrees of change.

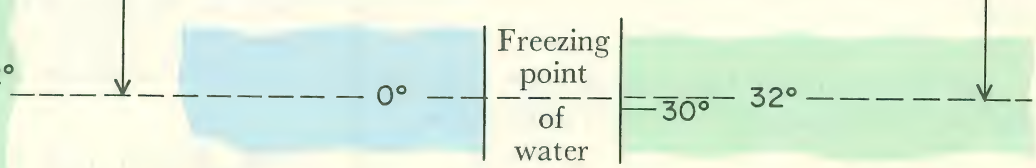
Ordinary unsalted water froze at 32° on his scale and boiled at 212°.

The *Fahrenheit scale* is still widely used in English-speaking countries.

In 1742, a Swedish astronomer, Anders Celsius, described another scale (a *centigrade scale*), placing 0° at the freezing point of water and 100° at the boiling point of water, both under normal barometric pressure at sea level.

With the general adoption of the metric system in the early 19th century, the centigrade scale was accepted by all but English-speaking countries.

Scientists universally favor the centigrade system of marking points on the thermometer.



You are familiar with the abbreviation for *degree*(°). Our abbreviation for readings *above 0°* is a raised plus sign ($^+10°$, $^+32°$); for readings *below 0°* we shall use a raised minus sign ($^-5°$, $^-11°$). Read "$^+10°$" as "positive ten degrees" or as "ten degrees above zero"; read "$^-5°$" as "negative five degrees" or as "five degrees below zero."

Using the scales on the left, please complete the following sentences.

A reading of $^+60°$ C is about the same as a reading of ______ F.

A reading of $^-18°$ C is about the same as a reading of ______ F.

A reading of ______ C is about the same as a reading of $^+22°$ F.

A reading of ______ C is about the same as a reading of $^-40°$ F.

Doing Some Computations with Changes

$^{-}5^\circ$ as a point and $^{-}5^\circ$ as a change

It's five degrees below zero: $^{-}5^\circ$. That is a point or a position on a temperature scale.

The temperature has fallen five degrees: $^{-}5^\circ$. That describes a change — giving the direction ($^{-}$) and the amount (5°).

For the remainder of this page, we shall use a $^{-}5^\circ$ to mean a drop of 5°, and a $^{+}5^\circ$ to mean a rise of 5°.

If the temperature goes up 10° and then up another 7°, it has gone up a total of 17°. We shall abbreviate this statement by writing:

$$^{+}10^\circ + {}^{+}7^\circ = {}^{+}17^\circ$$

Read it as "positive 10° plus positive 7° equals positive 17°."

If the temperature goes up 3° and then falls 8°, we shall indicate that by:

$$^{+}3^\circ + {}^{-}8^\circ = \_\_\_\_\_$$

Read this as "positive 3° plus negative 8° equals negative 5°."

Here are a few examples you can use to check whether you understand this way of showing the total change in temperature that results when a first change is followed by a second change.

1. $^{+}3^\circ + {}^{+}7^\circ = \_\_\_\_\_^\circ$	5. $^{+}18^\circ + {}^{+}14^\circ = \_\_\_\_\_^\circ$
2. $^{-}3^\circ + {}^{+}7^\circ = \_\_\_\_\_^\circ$	6. $^{+}18^\circ + {}^{-}14^\circ = \_\_\_\_\_^\circ$
3. $^{+}3^\circ + {}^{-}7^\circ = \_\_\_\_\_^\circ$	7. $^{-}18^\circ + {}^{+}14^\circ = \_\_\_\_\_^\circ$
4. $^{-}3^\circ + {}^{-}7^\circ = \_\_\_\_\_^\circ$	8. $^{-}18^\circ + {}^{-}14^\circ = \_\_\_\_\_^\circ$
9. $0^\circ + {}^{+}9^\circ = \_\_\_\_\_^\circ$	13. $^{+}3.5^\circ + {}^{+}.8^\circ = \_\_\_\_\_^\circ$
10. $0^\circ + {}^{-}9^\circ = \_\_\_\_\_^\circ$	14. $^{-}3.5^\circ + {}^{+}.8^\circ = \_\_\_\_\_^\circ$
11. $^{+}17^\circ + 0^\circ = \_\_\_\_\_^\circ$	15. $^{-}3.5^\circ + {}^{-}.8^\circ = \_\_\_\_\_^\circ$
12. $^{-}17^\circ + 0^\circ = \_\_\_\_\_^\circ$	16. $^{+}3.5^\circ + {}^{-}.8^\circ = \_\_\_\_\_^\circ$

We are using an *addition sign* as a short way of saying:

followed by a change of

We shall use a *subtraction sign* as a short way of saying:

followed by the opposite of a change of

The *opposite of a rise of* 5° ($^{+}5^\circ$) is *a fall of* 5° ($^{-}5^\circ$).

$$^{+}7^\circ - {}^{+}5^\circ = {}^{+}2^\circ$$

The *opposite of a fall of* 8° ($^{-}8^\circ$) is *a rise of* 8° ($^{+}8^\circ$).

$$^{+}3^\circ - {}^{-}8^\circ = {}^{+}11^\circ$$

Here are a few examples to check your understanding. (Let's agree that we shall consider the opposite of a change of 0° to be 0°.)

17. $^{+}4^\circ - {}^{+}5^\circ = \_\_\_\_\_^\circ$	21. $0^\circ - {}^{+}16^\circ = \_\_\_\_\_^\circ$
18. $^{-}4^\circ - {}^{+}5^\circ = \_\_\_\_\_^\circ$	22. $0^\circ - {}^{-}16^\circ = \_\_\_\_\_^\circ$
19. $^{+}4^\circ - {}^{-}5^\circ = \_\_\_\_\_^\circ$	23. $^{+}16^\circ - 0^\circ = \_\_\_\_\_^\circ$
20. $^{-}4^\circ - {}^{-}5^\circ = \_\_\_\_\_^\circ$	24. $^{-}16^\circ - 0^\circ = \_\_\_\_\_^\circ$

Suppose that a temperature went through three successive changes — each being a change of $^{-}5^\circ$. We shall use a *multiplication sign* to show successive changes of the same size and direction. Thus:

$$^{-}5^\circ \times 3 = {}^{-}15^\circ \quad \text{or} \quad 3 \times {}^{-}5^\circ = {}^{-}15^\circ$$

Read this as "negative 5° multiplied by 3 equals negative 15°, or 3 times negative 5° equals negative 15°."

Please do the following examples.

25. $^{+}7^\circ \times 3 = {}^{+}21^\circ$	27. $3 \times {}^{+}7^\circ = \_\_\_\_\_^\circ$
26. $^{-}7^\circ \times 3 = {}^{-}21^\circ$	28. $3 \times {}^{-}7^\circ = \_\_\_\_\_^\circ$
29. $^{+}9^\circ \times 5 = \_\_\_\_\_^\circ$	33. $10 \times {}^{-}8^\circ = \_\_\_\_\_^\circ$
30. $^{-}9^\circ \times 5 = \_\_\_\_\_^\circ$	34. $10 \times {}^{+}8^\circ = \_\_\_\_\_^\circ$
31. $9 \times {}^{-}5^\circ = \_\_\_\_\_^\circ$	35. $^{+}10^\circ \times 8 = \_\_\_\_\_^\circ$
32. $9 \times {}^{+}5^\circ = \_\_\_\_\_^\circ$	36. $^{-}10^\circ \times 8 = \_\_\_\_\_^\circ$

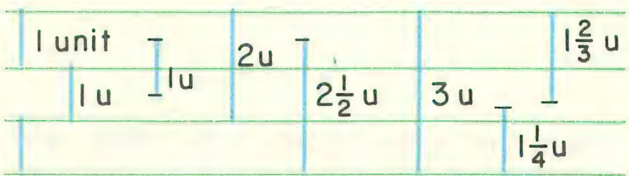
Because this arithmetic we use to compute total changes will keep coming up, let's take a further look at it.

A General Method

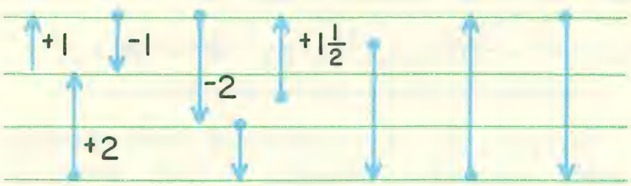
We often need to describe changes and whole sequences of changes — changes in temperature, changes in position, changes in a bank account, changes in weight, changes in scores, changes in height, changes in time, etc.

Let's develop a system we can adapt to meet any needs of this kind.

First, we shall select a convenient way to indicate a unit of change. Let the unit be the distance between two successive green lines.

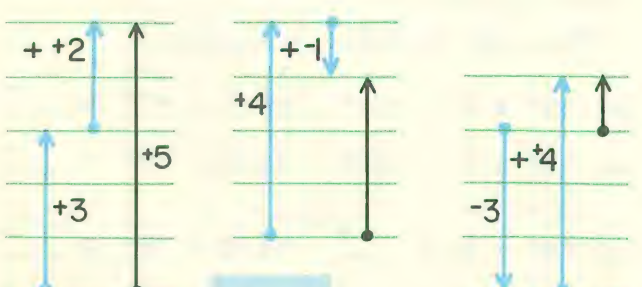


Next, we shall say that a unit of change in the up direction is a *positive* unit of change, and in the opposite direction, a *negative* unit of change. *(We left four changes for you to label. Please do so.)*



ADDITION

We shall think of one change followed by another as *adding* the second change, and the sum of the changes as the *net change*. Here's a sketch of the idea:



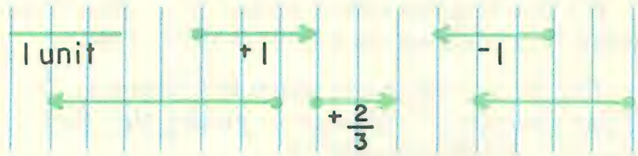
$^{+}4 + ^{-}1 =$ ____

$^{+}3 + ^{+}2 = ^{+}5$

$^{-}3 + ^{+}4 =$ ____

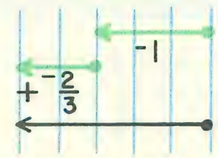
$$\underbrace{\text{positive 3}}_{\text{first change}}\ \underbrace{\text{plus positive 2}}_{\text{followed by second change}} = \underbrace{\text{positive 5}}_{\text{net change}}$$

We can, of course, change our unit and sketch. Let us say that our unit is the horizontal distance as shown, and a positive change is shown as moving from left to right. *(Please label the arrows.)*

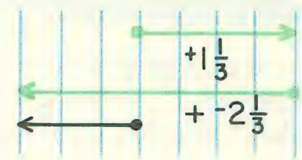


A change followed by another.

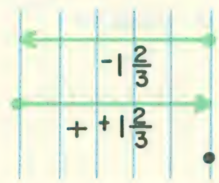
(Please complete the sentences and label the arrows.)



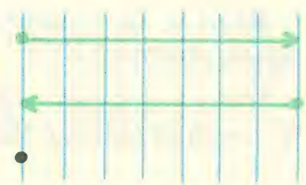
$^{-}1 + ^{-}\frac{2}{3} =$ ____



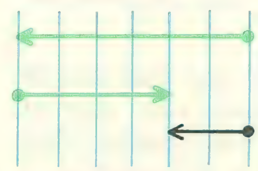
$^{+}1\frac{1}{3} + ^{-}2\frac{1}{3} =$ ____



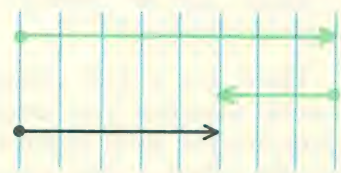
$^{-}1\frac{2}{3} + ^{+}1\frac{2}{3} =$ ____



$^{+}2\frac{1}{3} +$ ____ $=$ ____



$+ =$ ____



$+ =$ ____

(Please draw labeled sketches for the following.)

$^{+}2\frac{1}{3} + ^{-}1\frac{2}{3} =$ ____

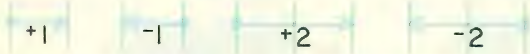
$^{-}1\frac{1}{3} + ^{-}\frac{2}{3} =$ ____

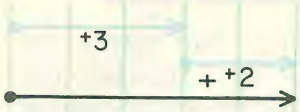
$^{+}1 + ^{-}2\frac{1}{3} =$ ____

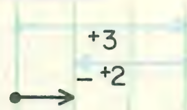
$^{-}2 + ^{+}1\frac{1}{3} =$ ____

SUBTRACTION

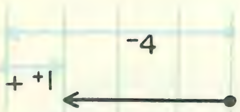
We shall think of one change followed by the *opposite* of a second change as *subtracting* the second change. So, to subtract a second change, just add its opposite. Here are sketches to suggest the idea. *(Please complete the sentences.)*

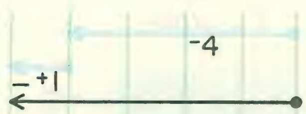




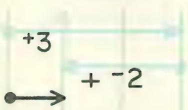


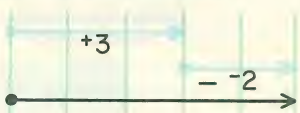
${}^{+}3 + {}^{+}2 =$ ____ ${}^{+}3 - {}^{+}2 =$ ____



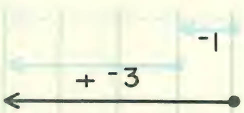


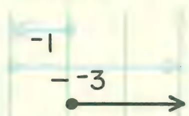
${}^{-}4 + {}^{+}1 =$ ____ ${}^{-}4 - {}^{+}1 =$ ____





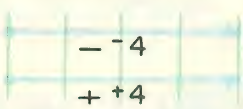
${}^{+}3 + {}^{-}2 =$ ____ ${}^{+}3 - {}^{-}2 =$ ____

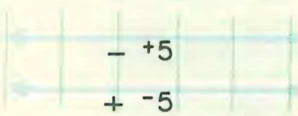




${}^{-}1 + {}^{-}3 =$ ____ ${}^{-}1 - {}^{-}3 =$ ____

We might summarize this way:





In general, for all numbers *a* and *n*,

$$
\begin{aligned}
{}^{+}a - {}^{-}n &= {}^{+}a + {}^{+}n\\
{}^{-}a - {}^{-}n &= {}^{-}a + {}^{+}n\\
{}^{+}a - {}^{+}n &= {}^{+}a + {}^{-}n\\
{}^{-}a - {}^{+}n &= {}^{-}a + {}^{-}n.
\end{aligned}
$$

Any problem in subtracting positive and negative numbers can be solved by doing a related problem in addition.

Yesterday, between 6 A.M. and noon, the temperature rose 3 degrees. Between noon and 6 P.M., it fell 9 degrees.

${}^{+}3 + {}^{-}9 =$ ____

It was ______ degrees ________________ (warmer or colder) at 6 P.M. yesterday than at 6 A.M.

Today, the rise in the morning was the same, but in the afternoon its change was just the opposite of the day before.

${}^{+}3 - {}^{-}9 =$ ____

It was ______ degrees ________________ (warmer or colder) at 6 P.M. today than at 6 A.M.

I had $15 more in savings than in bills. The postman brought me a bill for $7.

${}^{+}15 + {}^{-}7 =$ ____

So, I now have $________ more in savings than in bills.

On a different date, I again had $15 more in savings than in bills. When the postman came, he explained that he should have delivered a bill of $7 to someone else, so he took the bill away.

${}^{+}15 - {}^{-}7 =$ ____

After he took away the bill, I had $________ more in savings than in bills.

At still another time, I found I had $10 more in bills than I had in savings.

When the postman came, he told me of another mistake. He had by mistake brought me a $2-check the day before, so I gave him back the $2-check.

${}^{-}10 - {}^{+}2 =$ ____

After he took the check back, I had $________ more in bills than in savings.

MULTIPLICATION

You know how to make sense out of the following statements in addition and subtraction of positive and negative numbers.

$$^{-}5 + {}^{+}3 = {}^{-}2$$

$$^{+}2 - {}^{-}4 = {}^{+}6$$

In the addition example, just think of the net change produced by a change of $^{-}5$ followed by a change of $^{+}3$. The net change is ______.

In the subtraction example, think of the net change produced by a change of $^{+}2$ followed by the opposite of a change of $^{-}4$. The net change is ______.

Now, how shall we make sense out of examples of multiplication of positive and negative numbers?

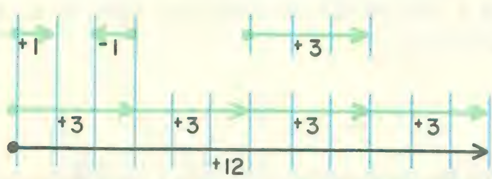
$$^{+}3 \times {}^{+}2 = ?$$

$$^{-}4 \times {}^{-}5 = ?$$

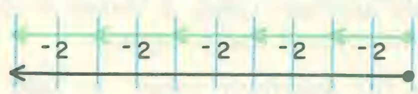
Let's build from what we already know. We can easily make sense out of one kind of multiplication example.

$^{+}3 \times 4 = ?$ $\quad$ $5 \times {}^{-}2 = ?$

A few sketches will suggest the idea:



$^{+}3 \times 4$ = ____ or $4 \times {}^{+}3$ = ____



$^{-}2 \times 5$ = ____ or $5 \times {}^{-}2$ = ____

You may draw sketches if you like, to help you complete the following:

$^{+}3 \times 8$ = ____ or $8 \times {}^{+}3$ = ____

$^{-}7 \times 5$ = ____ or $5 \times {}^{-}7$ = ____

$^{+}12 \times \frac{1}{2}$ = ____ or $\frac{1}{2} \times {}^{+}12$ = ____

$^{-}9 \times \frac{2}{3}$ = ____ or $\frac{2}{3} \times {}^{-}9$ = ____

$^{-}81 \times \frac{1}{3}$ = ____ or $\frac{1}{3} \times {}^{-}81$ = ____

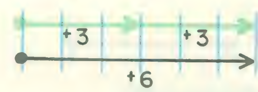
$^{+}36 \times \frac{1}{4}$ = ____ or $\frac{1}{4} \times {}^{+}36$ = ____

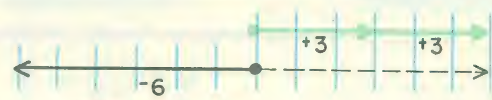
Let's introduce a new idea. Not only do we wish to multiply a change by a number, but we also wish to signal a change of direction of the result. We might use a small *CD* as a signal to change direction, and a small *NC* as a signal for no change in direction.

Examples.

$^{+}3 \times {}^{NC}2 = {}^{+}6$

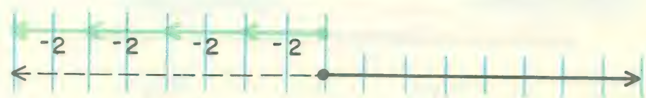
no change





$^{+}3 \times {}^{CD}2 = {}^{-}6$

change direction



$^{-}2 \times {}^{CD}4$ = ____

"Negative two multiplied by four . . . and change direction of the result."

Please complete the following sentences. (Draw sketches if you need them to help you.)

$^{-}4 \times {}^{CD}3$ = ____ or ${}^{CD}3 \times {}^{-}4$ = ____

$^{-}3 \times {}^{CD}4$ = ____ or ${}^{CD}4 \times {}^{-}3$ = ____

$^{+}9 \times {}^{NC}8$ = ____ or ${}^{NC}8 \times {}^{+}9$ = ____

$^{+}8 \times {}^{NC}9$ = ____ or ${}^{NC}9 \times {}^{+}8$ = ____

$^{+}7 \times {}^{CD}4$ = ____ or ${}^{CD}4 \times {}^{+}7$ = ____

$^{+}4 \times {}^{CD}7$ = ____ or ${}^{CD}7 \times {}^{+}4$ = ____

Simplifying the Shorthand

Do we need all of the following signs?

+   −   NC   CD

Mathematicians use the $^{+}$ instead of the ^{NC} and the $^{-}$ instead of the ^{CD}. Look through the twelve examples at the end of the previous page. Mathematicians would write them like this:

$${}^{-}4 \times {}^{-}3 = {}^{-}3 \times {}^{-}4 = {}^{-}3 \times {}^{-}4 = {}^{-}4 \times {}^{-}3 = {}^{+}12$$

$${}^{+}9 \times {}^{+}8 = {}^{+}8 \times {}^{+}9 = {}^{+}8 \times {}^{+}9 = {}^{+}9 \times {}^{+}8 = {}^{+}72$$

$${}^{+}7 \times {}^{-}4 = {}^{-}4 \times {}^{+}7 = {}^{+}4 \times {}^{-}7 = {}^{-}7 \times {}^{+}4 = {}^{-}28$$

We might have lost a little information because we can no longer tell which is the multiplier.

$${}^{-}2 \times {}^{NC}3 = {}^{NC}3 \times {}^{-}2 = {}^{+}3 \times {}^{-}2 = {}^{-}6$$

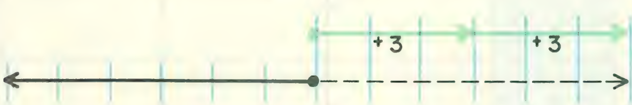
In the first two expressions, we are taking a change of $^{-}2$ three times and the result, $^{-}6$, is not changed in direction.

The third expression could mean just what the first two do, but it could also mean a change of $^{+}3$ taken twice and the result changed in direction.

Let's illustrate both ideas with sketches:

${}^{-}2 \times {}^{NC}3 =$ ____

and



${}^{+}3 \times {}^{-}2 =$ ____

The results are alike. So we can indeed use our *NC* and *CD* ideas to make sense out of the mathematician's way of doing multiplication with positive and negative numbers.

This situation is similar to the one in which we try to make sense out of the statement that $3 \times 2 = 6$. This could be thought of as saying that

$$\$3 \times 2 = \$6 \quad \text{or} \quad 3 \times \$2 = \$6.$$

Both ways of looking at "$3 \times 2 = 6$" make sense.

More Simplification

Let's agree that we shall sometimes omit the $^{+}$ in our computations with directed numbers. We shall act just as if it had been written. So, for positive numbers like $^{+}3$ and $^{+}7$, we shall simply write 3 and 7. We shall write negative numbers as we have always done.

Does this cause any confusion in multiplication examples?

${}^{+}12 \times {}^{-}4$	=	$12 \times {}^{-}4$	=	${}^{-}48$
${}^{+}9 \times {}^{+}7$	=	9×7	=	____
${}^{-}6 \times {}^{+}8$	=		=	____
${}^{-}10 \times {}^{-}10$	=		=	____
${}^{+}8 \times {}^{+}5$	=		=	____
${}^{-}8 \times {}^{-}5$	=		=	____

Does our agreement cause any confusion in addition and subtraction examples?

$3 + {}^{-}4$	= ____	${}^{-}7 + {}^{-}3$	= ____
$5 - 7$	= ____	$\quad + 8$	= 17
$5 - {}^{-}7$	= ____	${}^{-}6 - \quad$	= 4
${}^{-}5 \times 7$	= ____	${}^{-}6 \times {}^{-}10$	= ____
$3 \times {}^{-}12$	= ____	$\quad \times 13$	= 65
$23 - {}^{-}18$	= ____	$23 - \quad$	= 5
${}^{-}18 - 23$	= ____	$18 - \quad$	= 41
${}^{-}24 \times \frac{1}{3}$	= ____	${}^{-}24 \times {}^{-}\frac{1}{3}$	= ____

Now let's think about division.

DIVISION

Division with positive and negative numbers is much like division with undirected numbers. *(Please complete the following examples.)*

$8 \div 4 = \underline{\ 2\ }$ because $\underline{\ 4\ } \times 2 = 8$

$12 \div 3 =$ ____ because ____ $\times\ 3 = 12$

$3 \div \frac{1}{2} =$ ____ because ____ $\times \frac{1}{2} = 3$

$21 \div 3 = ($ ____ $\times\ 3) \div 3 =$ ____

Dividing by an undirected number "undoes" the result of multiplying by it. This is precisely what dividing by a directed number does.

${}^{+}10 \div {}^{-}5 = \underline{{}^{-}2}$ because $\underline{{}^{-}2} \times {}^{-}5 = {}^{+}10$

${}^{-}8 \div {}^{-}2 =$ ____ because $\underline{{}^{+}4} \times {}^{-}2 = {}^{-}8$

${}^{+}6 \div {}^{+}3 =$ ____ because ____ $\times\ {}^{+}3 = {}^{+}6$

${}^{-}10 \div {}^{+}2 =$ ____ because ____ $\times$ ____ $=$ ____

${}^{-}21 \div {}^{+}3 = ($ ____ $\times\ {}^{+}3) \div {}^{+}3 =$ ____

To solve this problem:

${}^{+}14 \div {}^{-}7 =$ ____

just solve this problem:

____ $\times\ {}^{-}7 = {}^{+}14$

Solve the following problems.

1. ${}^{-}30 \div {}^{+}5 =$ ____
2. ${}^{+}12 \div {}^{-}3 =$ ____
3. ${}^{-}18 \div {}^{-}9 =$ ____
4. ${}^{+}20 \div {}^{+}4 =$ ____
5. $25 \div {}^{-}5 =$ ____
6. ${}^{-}30 \div 6 =$ ____
7. ${}^{-}56 \div 8 =$ ____
8. ${}^{-}56 \div {}^{-}8 =$ ____
9. $15 \div {}^{-}3 =$ ____
10. $15 \div {}^{-}\frac{1}{3} =$ ____
11. $0 \div {}^{-}2 =$ ____
12. ____ $\times\ {}^{-}2 = 0$
13. ${}^{-}\frac{1}{2} \div {}^{-}\frac{1}{2} =$ ____
14. ${}^{-}\frac{1}{3} \div \frac{1}{3} =$ ____
15. ${}^{-}\frac{3}{2} \div {}^{-}\frac{3}{2} =$ ____
16. $\frac{1}{3} \div {}^{-}\frac{1}{3} =$ ____

SUMMARY

Complete the following table to show the results of adding, subtracting, multiplying, and dividing with positive and negative numbers.

	if $a < b$	if $a = b$	if $a > b$
${}^{+}a + {}^{+}b$	positive		
${}^{+}a + {}^{-}b$	negative	0	
${}^{-}a + {}^{+}b$			
${}^{-}a + {}^{-}b$			
${}^{+}a - {}^{+}b$			positive
${}^{+}a - {}^{-}b$		positive	
${}^{-}a - {}^{+}b$			
${}^{-}a - {}^{-}b$			
${}^{+}a \times {}^{+}b$	positive		
${}^{+}a \times {}^{-}b$			
${}^{-}a \times {}^{+}b$			
${}^{-}a \times {}^{-}b$			
${}^{+}a \div {}^{+}b$			
${}^{+}a \div {}^{-}b$			
${}^{-}a \div {}^{+}b$			
${}^{-}a \div {}^{-}b$			

I. In the green-tinted blocks, write the sum of the numbers in the blue-tinted blocks that occur in the same row and column.

ADDITION

				+4				
				+3				
				+2				
				+1				
-4	-3	-2	-1		+1	+2	+3	+4
				-1				
				-2				
				-3				
				-4				

II. In the green-tinted blocks, write the product of the numbers in the blue-tinted blocks that occur in the same row and column.

MULTIPLICATION

				+4				
				+3				
				+2				
				+1				
-4	-3	-2	-1		+1	+2	+3	+4
				-1				
				-2				
				-3				
				-4				

III. Pattern hunts.

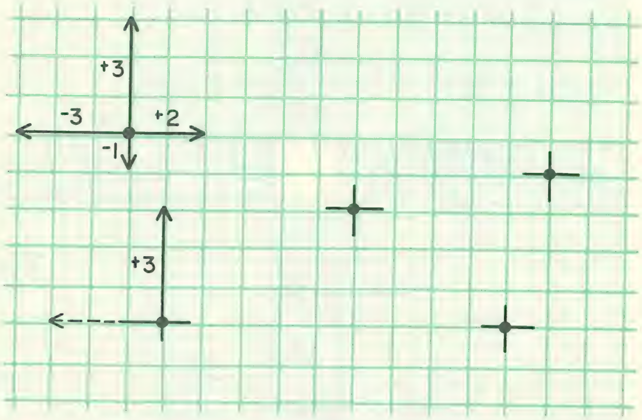
1)	2)	3)	4)	5)
10 → 17	-8 → -16	7 → 15	7 → -13	6 → -36
-3 → 4	.7 → ___	4 → ___	4 → ___	-3 → -9
0 → ___	-90 → ___	1 → ___	1 → ___	3 → ___
-10 → ___	___ → 0	-2 → ___	-2 → ___	-6 → ___
___ → 0	___ → -.5	-5 → ___	-5 → ___	___ → 0
$-9\frac{7}{8}$ → ___	$-\frac{1}{6}$ → ___	-8 → ___	-8 → ___	4 → ___
___ → $1\frac{1}{2}$	___ → $\frac{3}{4}$	-11 → ___	-11 → ___	___ → -16
$a \rightarrow a+7$	$n \rightarrow 2n$	$x \rightarrow 2x+1$	$x \rightarrow -2x+1$	$m \rightarrow m^2 \times (-1)$

6)	7)	8)	9)	10)
5 → 30	-4 → 20	70 → 30	3 → 15	4 → ___
-5 → 20	-3 → ___	30 → ___	-2.5 → ___	3 → ___
-4 → ___	___ → 14	60 → ___	4 → ___	2 → ___
4 → ___	___ → 11	___ → 60	-3.5 → ___	1 → ___
-6 → ___	___ → 8	120 → ___	5 → ___	0 → ___
6 → ___	___ → 2	-20 → ___	-4.5 → ___	-1 → ___
___ → 6	___ → -1	-50 → ___	7 → ___	-2 → ___
___ → 6	___ → -7	___ → -50	___ → 91	-3 → ___
$y \rightarrow y^2 + y$	$c \rightarrow -3c+8$	$p \rightarrow 100-p$	$q \rightarrow 2q^2-q$	$r \rightarrow r^2+2r+1$

Horizontal and Vertical Changes

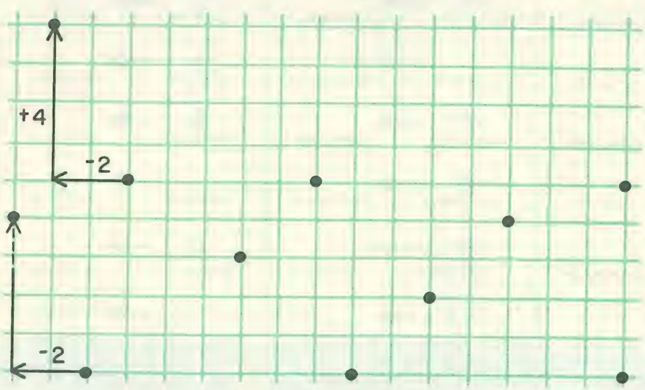
I. We wish to indicate four changes of position for each of the five points shown below. You can get the rules without any explanation.

Horizontal: $^{-}3$, $^{+}2$ Vertical: $^{+}3$, $^{-}1$

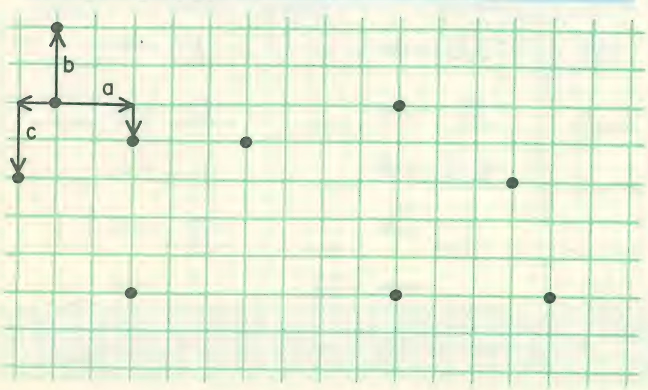


II. We wish to indicate a horizontal change followed by a vertical change. Do you get our rules?

Horizontal: $^{-}2$ followed by Vertical: $^{+}4$

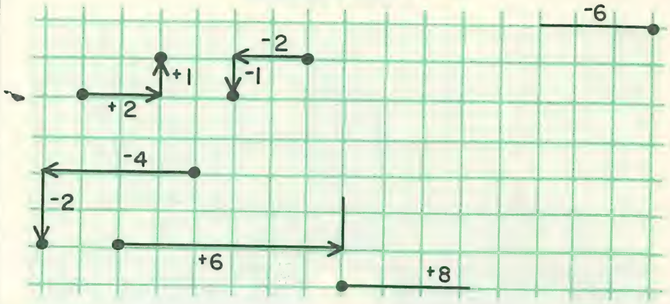


III. Change a: (H: $^{+}2$, V: $^{-}1$)
Change b: (H: 0, V: $^{+}2$)
Change c: (H: $^{-}1$, V: $^{-}2$)



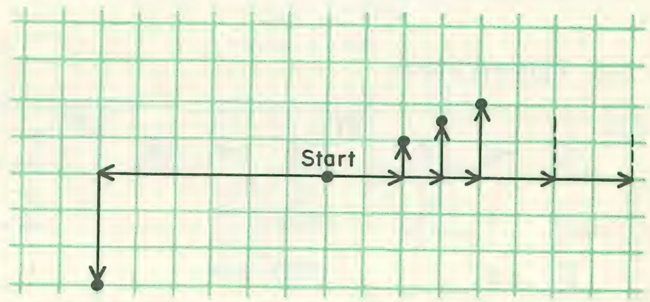
IV. A horizontal change followed by a vertical change that is half the horizontal change.

$$V = \frac{1}{2}H$$



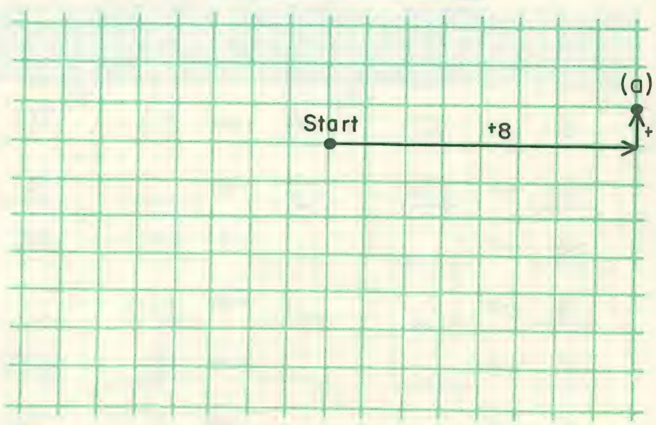
V. Pairs of horizontal followed by vertical changes, all starting from the same point. ($^{+}2$, $^{+}1$), ($^{+}3$, $^{+}1.5$), etc. *(Find at least four more new positions.)*

$$V = \frac{1}{2}H$$



VI. Each vertical change is $^{+}3$ less than half the corresponding horizontal change — all from the same point.

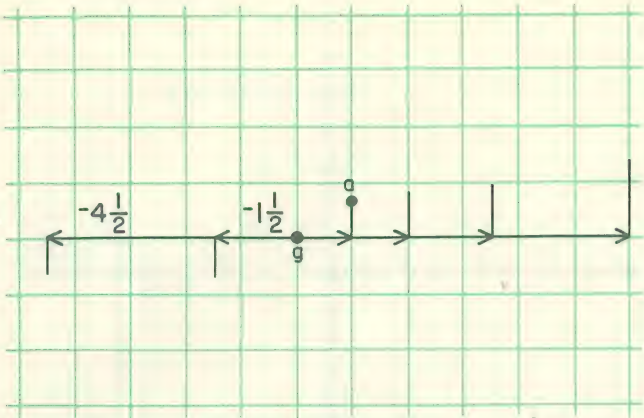
$$V = \frac{1}{2}H - {}^{+}3$$



$V = \frac{1}{2}H - {}^{+}3$

	a	b	c	d	e	f
If H is	+8	+6	0	+4	+3	-3
then V is	+1	0				

I. $V = \frac{2}{3} H$

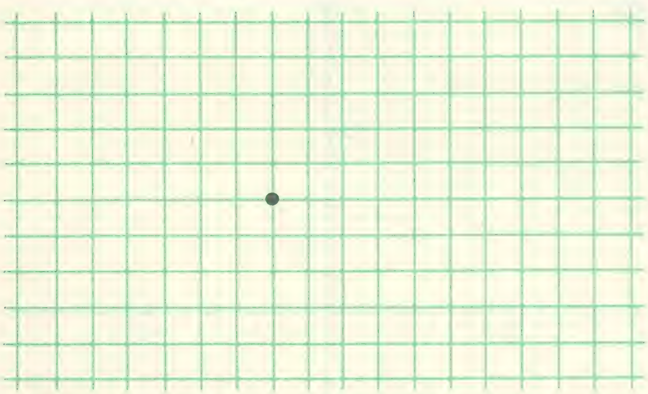


	a	b	c	d	e	f	g
If H is	$^+1$	$^+2$	$^+3\frac{1}{2}$	$^-1\frac{1}{2}$	$^+6$	$^-4\frac{1}{2}$	0
then V is ...	$^+\frac{2}{3}$						

Draw the line that would include all the points you have located.

II. From a single point, locate many points by following the rules given. You may wish to complete the tables below as you work on the graphs. *(Omit the arrows if possible.)*

$V = H$ $\quad$ $V = \frac{1}{2} H$ $\quad$ $V = \frac{1}{4} H$



$V = H$

H	$^+1$	$^+2$	$^+3$	$^-1$	$^-3$	0	$\frac{^-1}{2}$	$^+.1$	$\frac{^-7}{9}$	$^+19$
V										

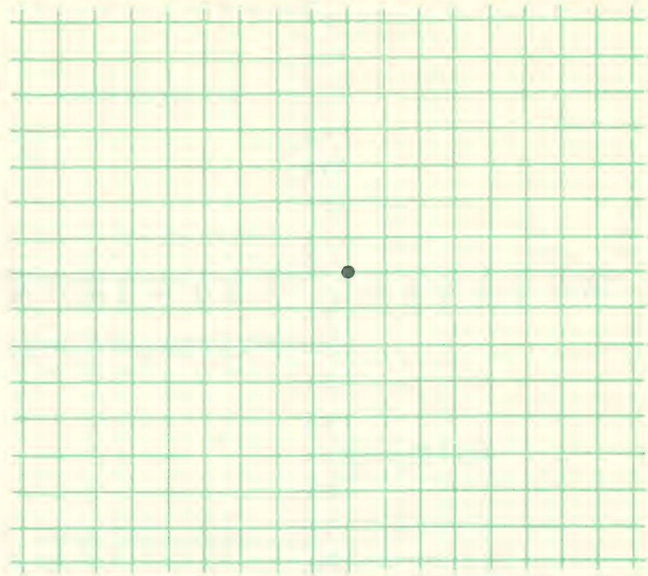
$V = \frac{1}{2} H$

H	$^+1$	$^+2$	$^-3$	$^-4$	0	$^+5$	$^+\frac{1}{2}$	$^+\frac{2}{3}$	$^-.8$	$^-1$
V										

$V = \frac{1}{4} H$

H	$^+1$	$^-1$	$^+2$	$^-2$	$^+\frac{1}{2}$	$\frac{^-1}{2}$	$^+.8$	$^-.8$	0	$^+\frac{2}{3}$
V										

III. $V = H$ $\quad$ $V = 2H$ $\quad$ $V = 3H$ $\quad$ $V = 4H$



$V = H$

H	$^+1$	$^+3$	$^-2$	
V	$^+1$	$^+3$		

$V = 2H$

H	$^-1$	$^+2$		
V				

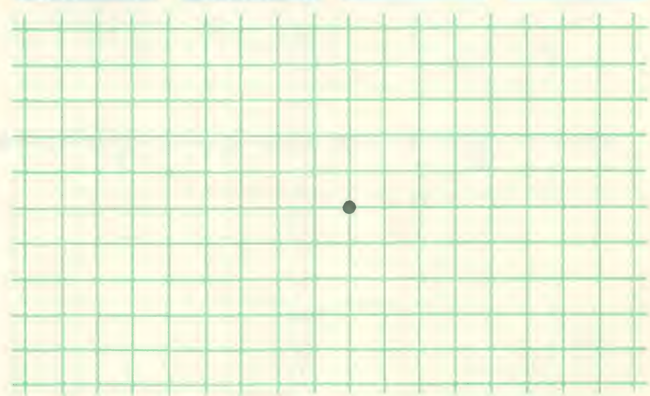
$V = 3H$

H	$\frac{^-1}{2}$	$^-1$		
V				

$V = 4H$

H	$^+1$	$^+\frac{1}{2}$		
V				

IV. $V = H \times {}^-1$ or $V = {}^-1 H$ $\quad$ $V = H \times {}^-2$ or $V = {}^-2 H$ $\quad$ $V = H \times {}^-3$ or $V = {}^-3 H$ $\quad$ $V = H \times \frac{^-1}{2}$ or $V = \frac{^-1}{2} H$



$V = {}^-1 H$

H				
V				

$V = {}^-2 H$

H				
V				

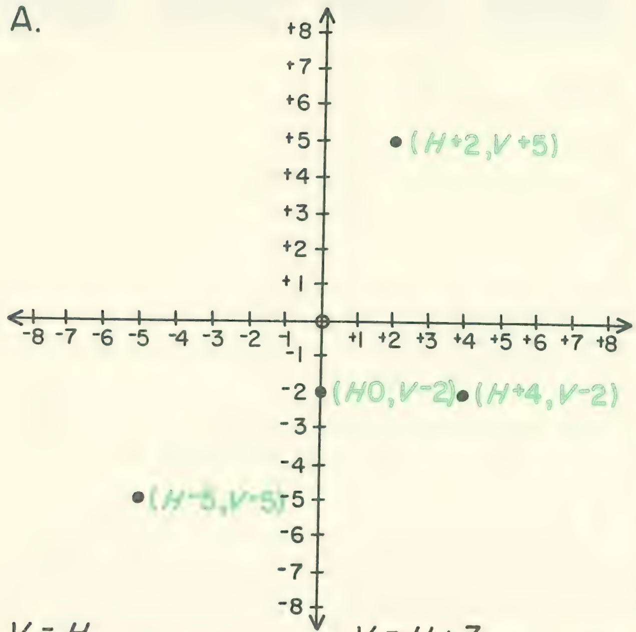
$V = {}^-3 H$

H				
V				

$V = \frac{^-1}{2} H$

H				
V				

A.



V = H

H	+3	-5		
V	+3			

V = H + 3

H	+2	-4		
V	+5			

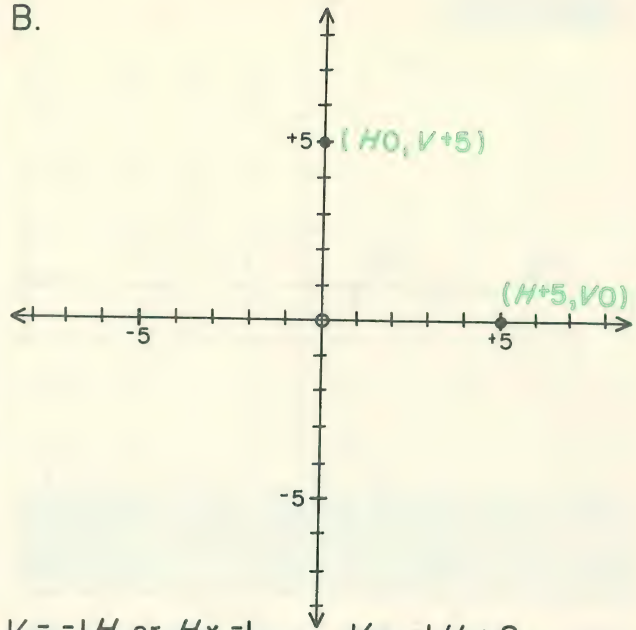
V = H − 2

H				2
V			-2	

V = H − 6

H				
V	0	-6		

B.



V = -1H or H x -1

H	+4	-3	0	-8
V	-4			

V = -1H + 2

H	0		+2	
V				

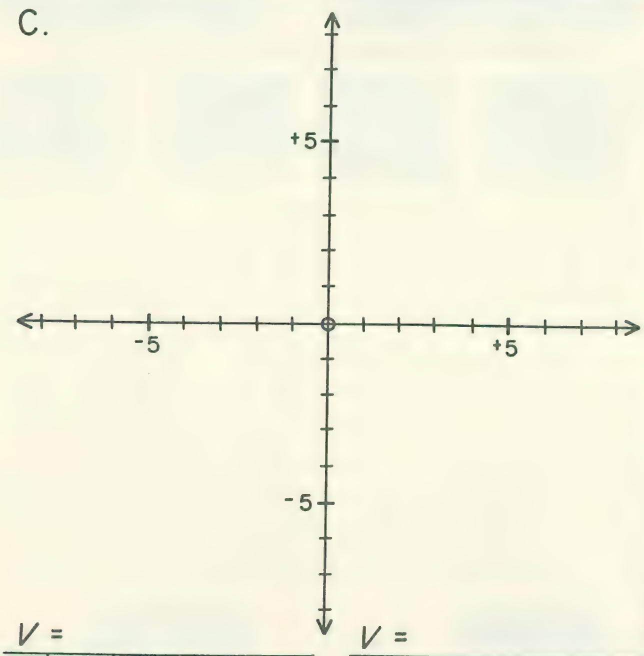
V = -1H − 3

H	+1	+3	-7	0
V				

V = -1H + 5

H	+8			
V				

C.



V =

H	1	4	-3	0
V	2	8	-6	

V =

H	0	2	-6	3
V	0	1	-3	

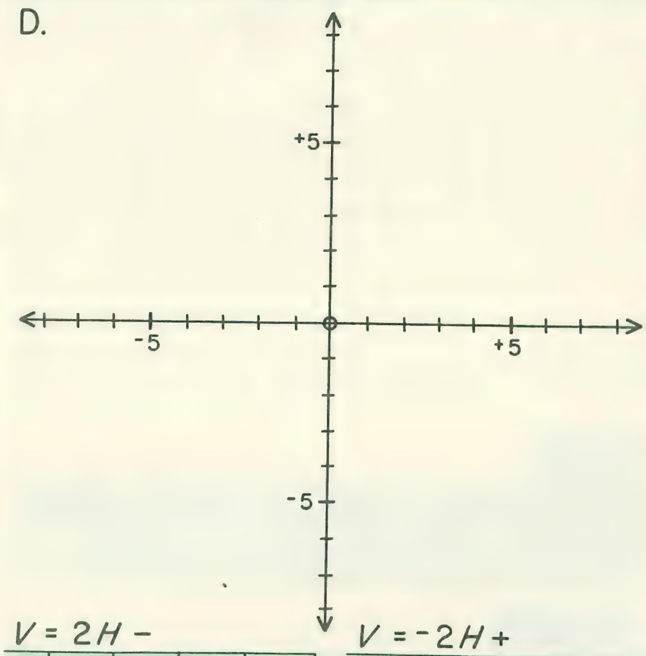
V = 2H − 2

H	-2	1	+4	0
V	-6			

V = $\frac{1}{2}$H +

H	3	0	-4	-8
V	$6\frac{1}{2}$			+1

D.



V = 2H −

H	-1	+3	+7	0
V	-8	0		-6

V = -2H +

H	+4		0	1
V	-6	+6	+2	0

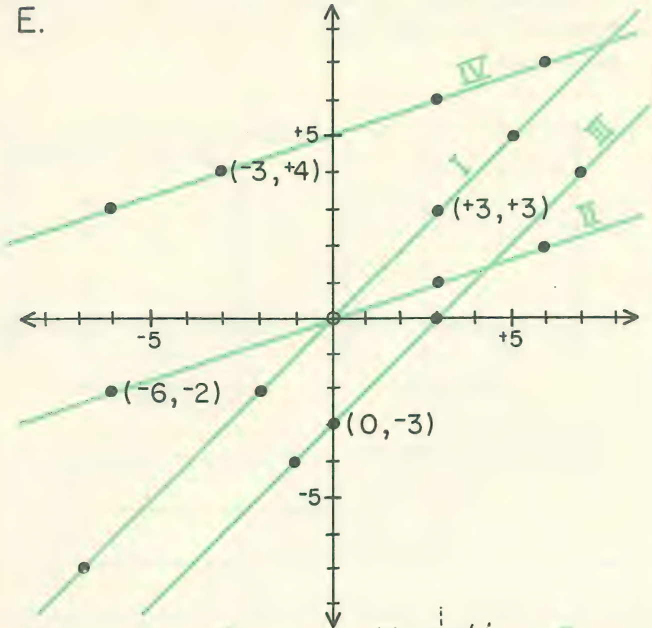
V =

H	6	0	-2	-6
V	0	-3	-4	-6

V = $-\frac{1}{2}$H −

H	0	8	-4	-2
V	-1		+1	0

E.



$V =$ $\frac{1}{3}H$ (II); $V =$ (I)

$V =$ I

H	+3	-2	+5	-7
V	+3			

$V = \frac{1}{3} H$ II

H	+3	+6	-6	0
V			-2	

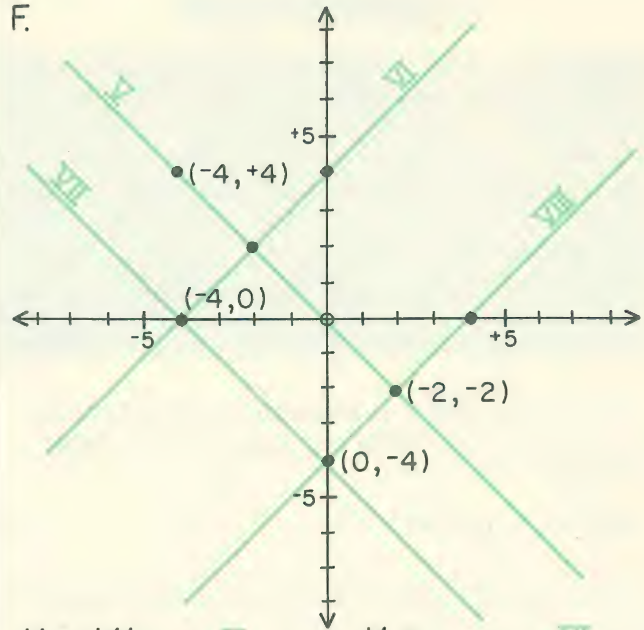
$V =$ III

H	+3	0	+7	-1
V		-3		

$V =$ IV

H	+6	-3		
V		+4		

F.



$V = -1H$ V

H	-4	+2		
V	+4	-2		

$V =$ VI

H	-2			
V	+2			

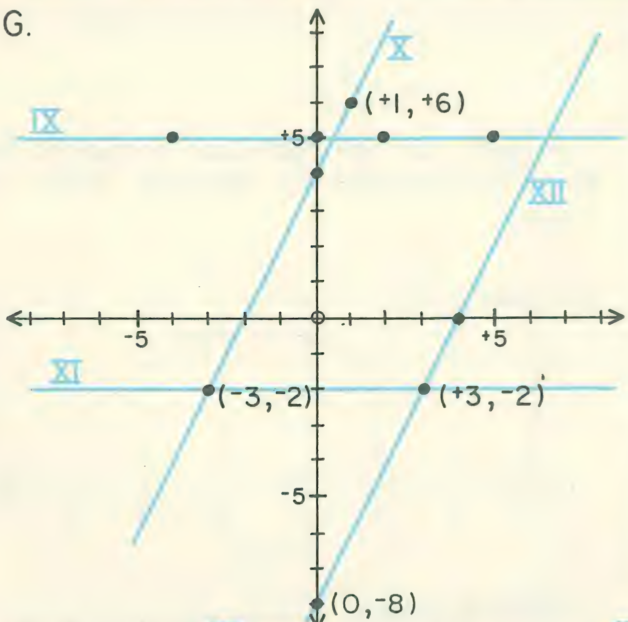
$V =$ VII

H	-4			
V	0			

$V =$ VIII

H	+2			
V	-2			

G.



$V = 0 \times H + 5$ IX

H	+2	0	-4	+5
V	+5		+5	

$V =$ X

H	+1			
V	+6			

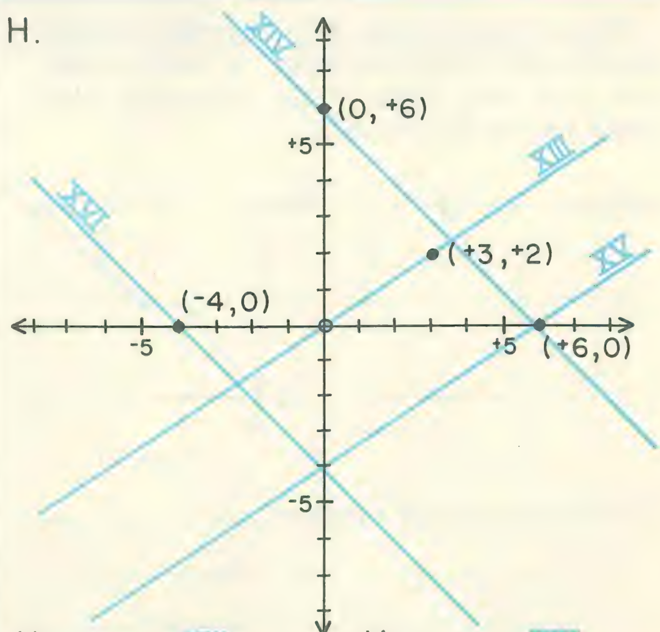
$V = 0 \times H -$ XI

H	+3	-3		
V	-2	-2		

$V =$ XII

H	+3			
V	-2			

H.



$V =$ XIII

H	+3			
V	+2			

$V =$ XIV

H	0			
V	+6			

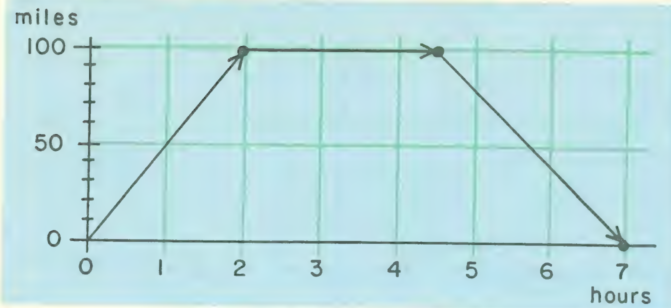
$V =$ XV

H	+6			
V	0			

$V =$ XVI

H	-4			
V	0			

Graphs of Trips



The graph above is a summary of distances traveled on a trip from our house to the shore and back.

What can you say?

The left-hand part of the graph evidently describes the trip to the shore. If the left-hand part had been either of the following, what could you say in each case?

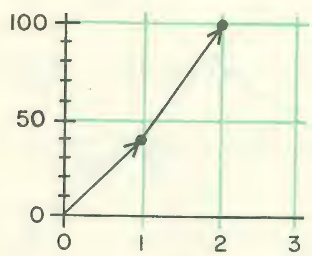


Figure I

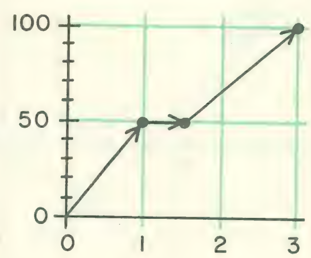


Figure II

What does *Figure I* report?

What does *Figure II* report?

Suppose that the center portion of the graph reported the following. What can you say in each case?

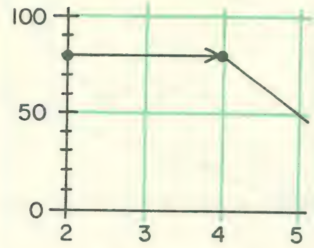


Figure III

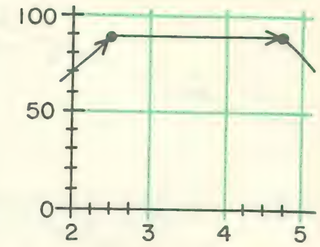


Figure IV

Figure III reports:

Figure IV reports:

Suppose that the right-hand portion of the graph had reported the following. What can you say?

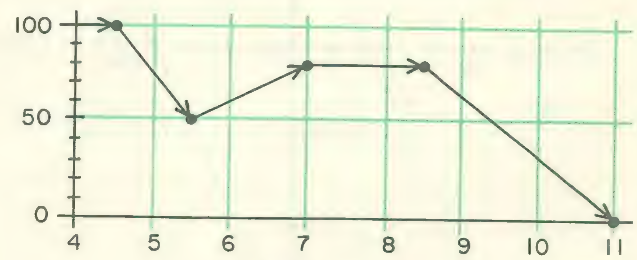


Figure V

Figure V reports:

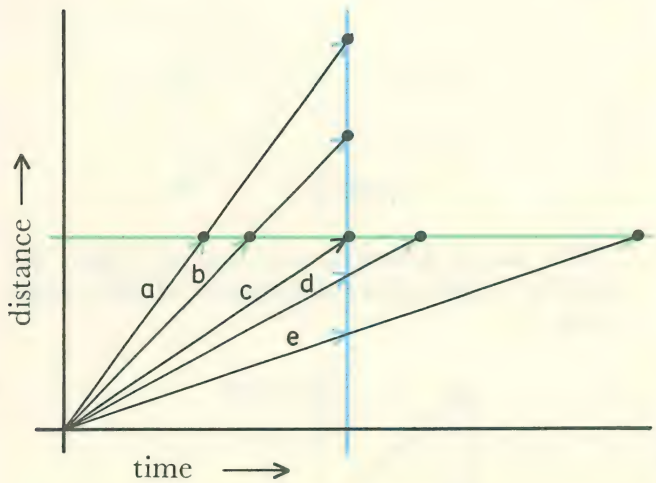
Comparing Speeds

Mr. Allen and Mr. Brown studied records to compare speeds of various cars.

Mr. Allen found out the time it took each to go a certain distance *(see green line)*.

Mr. Brown found out the distance each could go in a certain time *(see blue line)*.

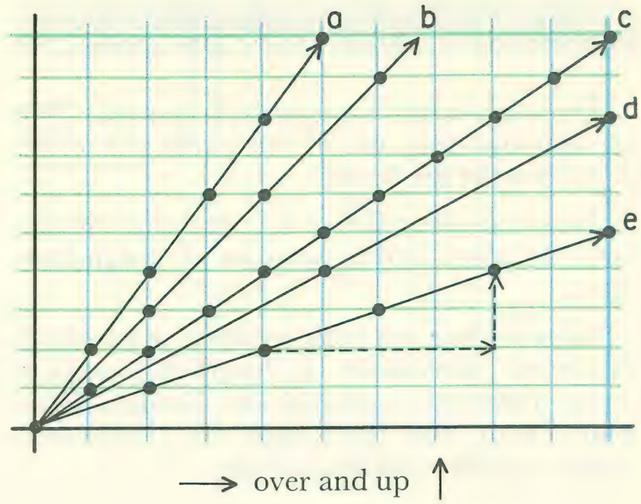
Here is a summary of their results shown in a single chart.



Car ______ was the fastest and car ______ was the slowest.

1. Car *a* was about ______ times as fast as car *e*.
2. Car *b* was about ______ times as fast as car *e*.
3. Car *b* was about ______ times as fast as car *d*.
4. Car *a* was about ______ times as fast as car *c*.
5. Car *c* was about ______ times as fast as car *d*.
6. Car *b* was about ______ times as fast as car *a*.

Let's take a more detailed look at the chart. We shall divide the distance into 10 equal distances and the time into ten equal periods of time.



We have marked heavy dots on each black line that passes through crossing points of the lines which indicate units of time and distance.

In each case, how many units must you go to the right and how many units up to move from one point to another? Record your findings:

	a	b	c	d	e
units up	4		3		2
units over		2		5	4

Try other points on the black lines to complete this record:

	a	b	c	d	e
units up		6	1		
units over	1			10	2

Try other points to complete this record:

	a	b	c	d	e
units up	8		7		
units over		6		5	6

Summary: Use the closest marked points you can in each case, and complete the following:

	(a)	(b)	(c)	(d)	(e)
units up	2	__	1	__	__
units over	1		1		

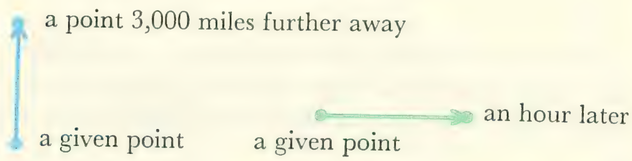
Look back to statements 1 through 6 in the previous column. Would your comments be as true of these five fractions as they were of the speeds? ______

Mixing Time and Space

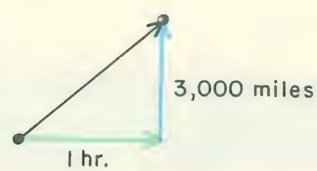
The captain of the space ship reported, "We are now traveling away from Earth at a speed of *3,000 miles per hour*."

That statement refers to a measure of distance in space and a measure of distance in time.

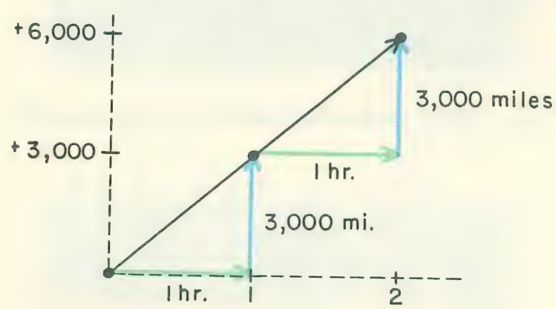
Suppose that we wish to draw a sketch of this event. We can let the length of a vertical arrow represent a change in distance away from Earth, and the length of a horizontal arrow represent a change in time.



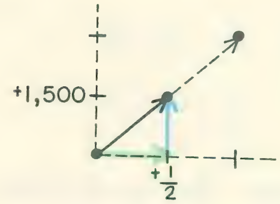
Let's combine these two measurements in this way:

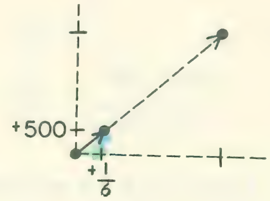


We shall say that the black arrow above suggests the idea that as an hour passed, the space ship traveled 3,000 miles. In two hours, this change would have happened twice.



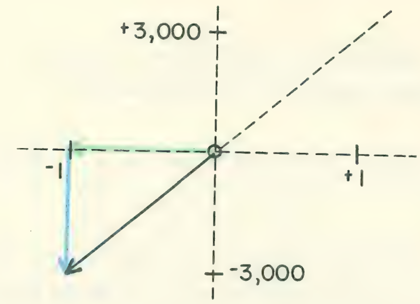
And we could indicate what would have happened in a half-hour, or in 10 minutes.





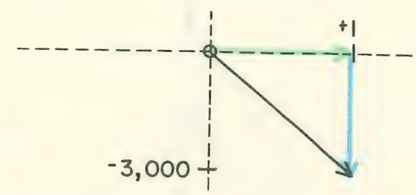
We can also extend our sketch to include some history of the flight.

Where were we an hour ago?

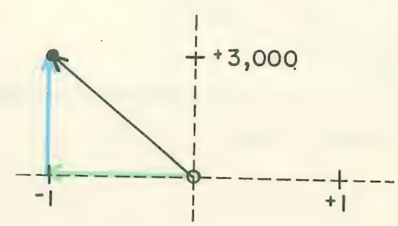


An hour ago ($^-1$ hr.) we were 3,000 miles less distant ($^-3,000$) from Earth.

If our return journey is at the same speed, as time passes the distance from Earth becomes less.

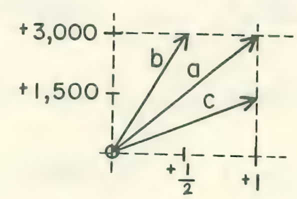


The sketch above shows that in 1 hour we shall be 3,000 miles less distant ($^-3,000$) from Earth.



The sketch above suggests the idea that ______________ *(an hour ago* or *an hour from now)* we ________ *(were* or *will be)* ________ miles ____________ *(further from* or *closer to)* Earth.

We are space ship *a*. Two other space ships, *b* and *c*, are the same distance we are from Earth and traveling in the same direction.

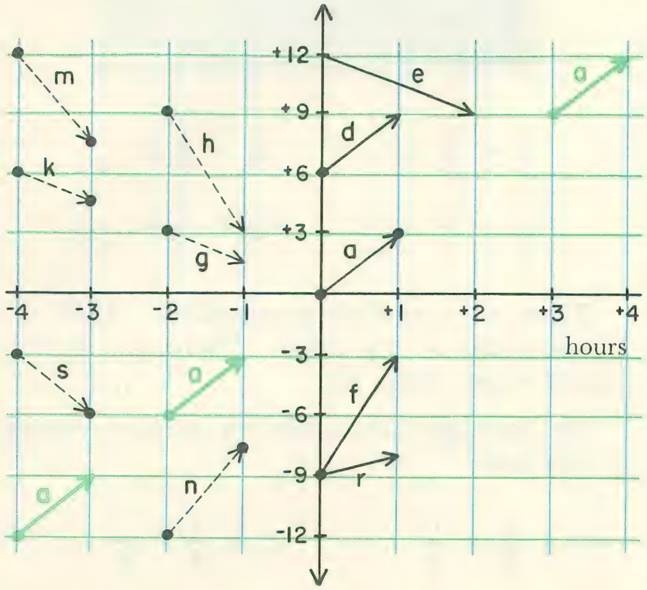


What information does the diagram suggest? Which space ship will go 3,000 miles in the least time? ______ Which will go the shortest distance in an hour? ______. If our speed in

(continued)

a is 3,000 mph then ship *b* must be traveling about ________ mph, and *c* about ________ mph.

In the control room of our space ship one of the crew plots some of the information he has about other space ships on the same route. (*The vertical axis shows distances in thousands of miles.*)



The broken-line arrows indicate information plotted 2 hours ago (*h, g, n*) and 4 hours ago (*m, k, s*). Distances from Earth are given in thousands of miles.

We are in space ship *a*. Other space ships going in the same direction as we are ________. Ships returning to Earth are ________.

The approximate speeds of the various space ships are:

a. ________ mph		*h.* ________ mph	
d. ________ mph		*k.* ________ mph	
e. ________ mph		*m.* ________ mph	
f. ________ mph		*n.* ________ mph	
g. ________ mph		*r.* ________ mph	

s. ________ mph

Have we met any space ships during the past 4 hours? If so, which ones? ________.

Which space ship did we pass more than 4 hours ago? ______

Which ship shall we meet in the next 4 hours? ______. Which ships will pass us during the next 4 hours? ________.

We met ______ less than a half-hour ago, and ______ a little less than an hour ago. It's been ______ hours since we met ship *s*.

At present, we are ________ miles away from *r*, and in 3 hours we shall be ________ miles from *r*.

There are evidently two space ships traveling close together. They are ______ and ______. Should we be able to see them at this time? ________.

Ship ______ is ________ miles ahead of us and traveling in the same direction and at the same speed.

What can you say about ships *f* and *r*?

____

What can you say about

m and *k*? ________________________________

____

____

d and *h*? ________________________________

____

____

e and *k*? ________________________________

____

____

the two fastest ships? ______________________

____

____

e, g, and *k*? ____________________________

____

____

____

____

Your authors disagree about what is the most helpful approach to considering

Division by a Fraction.

We call our different approaches by the following names:

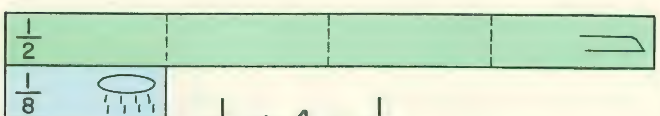
I. Sketches and Common Sense
II. Testing a Theory
III. Undoing Multiplication
IV. Mathematical Reasoning

Rather than selecting one of these approaches as better than the others, we are going to outline all of them for you — and you can decide for yourself which one is most helpful.

However, your authors are in complete agreement about this: that consideration of all four approaches is much better than concentrating on any one by itself.

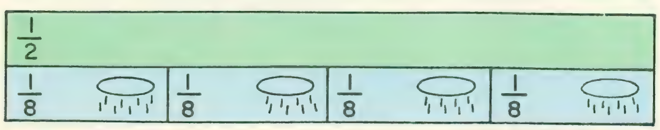
Approach I: Sketches and Common Sense

An ancient Egyptian (see page 33) might ask, "If I took a stick $\frac{1}{2}$ unit long and divided it by 4, how long would each of the pieces be?"



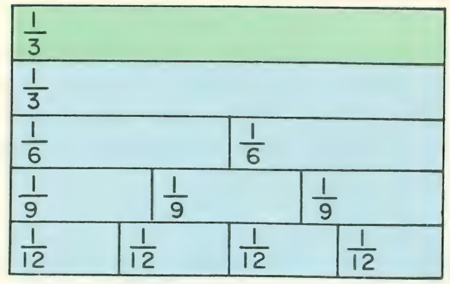
$$\frac{1}{2} \div 4 = \frac{1}{8}$$

Or, he might ask, "How many $\frac{1}{8}$ blocks does it take to make a train as long as a $\frac{1}{2}$ block?"



$$\frac{1}{2} \div \frac{1}{8} = \_\_\_\_$$

Suppose that an Egyptian carried out a series of experiments to see how many of his smaller blocks it would take to make trains each as long as the $\frac{1}{3}$ unit block. Suppose that the ones he found to fit were:



$\frac{1}{3} \div \frac{1}{3} = 1$ $\qquad$ $\frac{1}{3} \div \frac{1}{9} = \_\_\_$

$\frac{1}{3} \div \frac{1}{6} = \_\_\_$ $\qquad$ $\frac{1}{3} \div \frac{1}{12} = \_\_\_$

There is 1 one-third in one-third. There are 2 one-sixths in one-third. There are 3 one-ninths in one-third. Etc.

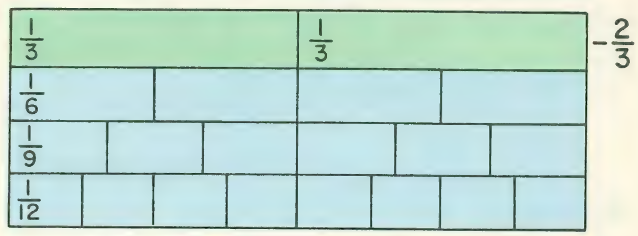
Or, he might have reported the same results in this way:

$1 \times \frac{1}{3} = \frac{1}{3}$ $\qquad$ $3 \times \frac{1}{9} = \frac{}{9} = \text{—}$

$2 \times \frac{1}{6} = \frac{2}{6} = \text{—}$ $\qquad$ $4 \times \frac{1}{12} = \text{—} = \text{—}$

To conserve space, let's use smaller sketches.

How many thirds, sixths, ninths, and twelfths are there in two-thirds?



In $\frac{2}{3}$ there are 4 sixths, 6 ninths, etc.

$\frac{2}{3} \div \frac{1}{6} = \_\_\_$ $\qquad$ $\frac{2}{3} \div \frac{1}{9} = \_\_\_$ $\qquad$ $\frac{2}{3} \div \frac{1}{12} = \_\_\_$

or

$\_\_\_ \times \frac{1}{6} = \frac{2}{3}$ $\qquad$ $\_\_\_ \times \frac{1}{9} = \frac{2}{3}$ $\qquad$ $\_\_\_ \times \frac{1}{12} = \frac{2}{3}$

Look at the sketch above. How many twelfths in $\frac{1}{6}$? How many in $\frac{3}{9}$? In $\frac{3}{6}$?

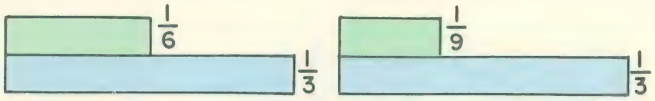
$\frac{1}{6} \div \frac{1}{12} = \_\_\_$ $\qquad$ $\frac{3}{9} \div \frac{1}{12} = \_\_\_$ $\qquad$ $\frac{3}{6} \div \frac{1}{12} = \_\_\_$

or

$\_\_\_ \times \frac{1}{12} = \frac{1}{6}$ $\qquad$ $\_\_\_ \times \frac{1}{12} = \frac{3}{9}$ $\qquad$ $\_\_\_ \times \frac{1}{12} = \frac{3}{6}$

How many thirds in one-sixth? How many thirds in one-ninth?

$\frac{1}{6} \div \frac{1}{3} = \_\_\_\_$ $\qquad$ $\frac{1}{9} \div \frac{1}{3} = \_\_\_\_$



Clearly, the answer in each case must be less than 1. If we think of our unit as 1 yard then

$$\frac{1}{3}\text{ yd.} = 12\text{ in. or } 1\text{ ft.};\ \frac{1}{6}\text{ yd} = \_\_\_\_\text{ in.};$$
$$\frac{1}{9}\text{ yd.} = \_\_\_\_\text{ in.}$$

How many feet in 6 inches? In 4 inches? Clearly, there's a half-foot in 6 inches and a third of a foot in 4 inches.

Let's try to restate each question asked about numbers as a question about some familiar and suitable system of measurement.

$2 \div \frac{1}{2} = \_\_\_$; $\frac{3}{4} \div \frac{1}{2} = \_\_\_$; $\frac{1}{4} \div \frac{1}{2} = \_\_\_$

Suppose that you think of money. How many half-dollars in two dollars? How many half-dollars in 75¢, or $\frac{3}{4}$ of a dollar? How many half-dollars in a quarter?

$1\frac{1}{2} \div \frac{1}{4} = \_\_\_$; $\frac{3}{4} \div \frac{1}{4} = \_\_\_$; $\frac{1}{3} \div \frac{1}{4} = \_\_\_$

How many quarter-dozens in a dozen and a half? Or, how many groups of 3 eggs in 18 eggs? How many groups of 3 eggs in 9 eggs? How many groups of 3 eggs in 4 eggs? Here is a picture of this last situation.

You can see that there is one group of 3 and a third of a group of three, or one and one-third groups of 3 eggs in 4 eggs.

$2 \div \frac{2}{7} = \_\_\_$; $\frac{6}{7} \div \frac{2}{7} = \_\_\_$; $\frac{1}{7} \div \frac{2}{7} = \_\_\_$

How many 2-day periods in 2 weeks? How many 2-day periods in 6 days? How many 2-day periods in 1 day?

$\frac{3}{4} \div \frac{1}{100} = \_\_\_$; $\frac{3}{10} \div \frac{1}{20} = \_\_\_$; $\frac{1}{10} \div \frac{1}{4} = \_\_\_$

How many pennies in 75¢? How many nickels in 3 dimes? How many quarters in a dime?

$\frac{1}{2} \div \frac{3}{16} = \_\_\_$; $\frac{1}{8} \div \frac{3}{16} = \_\_\_$; $\frac{3}{4} \div 1\frac{1}{2} = \_\_\_$

How many 3-oz. packages in half a pound? How many 3-oz. packages in 2 oz.? How many pound-and-a-halves in $\frac{3}{4}$ of a pound? How many 24-oz. packages can be made from 12 oz.?

$2\frac{2}{3} \div \frac{2}{3} = \_\_\_$; $1 \div \frac{2}{3} = \_\_\_$; $\frac{7}{8} \div \frac{2}{3} = \_\_\_$

How many 2-ft. lengths in 2 yards and 2 feet? How many 2-ft. strips in a yard?

Since $\frac{7}{8}$ of a yard is not best suited to help think about the third problem, let's talk about $\frac{7}{8}$ of a 24-hour day. How many $\frac{2}{3}$-of-a-day (16 hours) are there in $\frac{7}{8}$ of a day (21 hours)? How many 16's are there in 21?

$\frac{3}{8} \div \frac{1}{4} = \_\_\_$; $\frac{3}{8} \div \frac{1}{2} = \_\_\_$; $\frac{3}{8} \div \frac{3}{4} = \_\_\_$

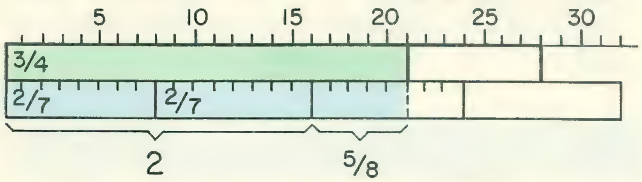
Write questions in which the same computing is involved as the mathematical statements above and below. Use familiar measurement situations such as ounces, hours, and inches.

$\frac{1}{4} \div \frac{1}{5} = \_\_\_$; $\frac{1}{5} \div \frac{3}{10} = \_\_\_$; $\frac{7}{20} \div \frac{7}{100} = \_\_\_$

But, suppose that you can't think of any familiar situations where the numbers used would come up naturally?

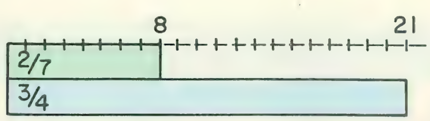
$\frac{3}{4} \div \frac{2}{7} = \_\_\_$; $\frac{2}{7} \div \frac{3}{4} = \_\_\_$

Invent a situation. Since 28 is the smallest number that is a multiple of both 4 and 7, start with a stick that's 28 units long. ($\frac{3}{4}$ of 28 is 21, and $\frac{2}{7}$ of 28 is 8.) How many chunks 8 units long are there in a chunk 21 units long?



There are $2\frac{5}{8}$ chunks each 8 units long in a 21-unit chunk. Or, one might say that there are $(21 \div 8)$ chunks.

How many 21-unit chunks are there in an 8-unit chunk; or, how much of a 21-unit chunk is there in an 8-unit chunk?



There are $8 \div 21$, or $\frac{8}{21}$, of a 21-unit chunk in an 8-unit chunk.

Use any method you wish to complete the following.

$\frac{2}{3} \div \frac{1}{9} =$	$\frac{3}{4} \div \frac{1}{8} =$	$1\frac{2}{5} \div \frac{1}{5} =$
$\frac{2}{3} \div \frac{2}{9} =$	$\frac{3}{4} \div \frac{1}{4} =$	$1\frac{2}{5} \div \frac{2}{5} =$
$\frac{2}{3} \div \frac{3}{9} =$	$\frac{3}{4} \div \frac{2}{4} =$	$1\frac{2}{5} \div \frac{1}{10} =$
$\frac{2}{3} \div \frac{4}{9} =$	$\frac{3}{4} \div \frac{3}{4} =$	$1\frac{2}{5} \div \frac{3}{10} =$
$\frac{2}{3} \div \frac{5}{9} =$	$\frac{3}{4} \div \frac{5}{4} =$	$\frac{5}{8} \div 2 =$
$\frac{2}{3} \div \frac{6}{9} =$	$\frac{3}{4} \div \frac{9}{4} =$	$\frac{5}{8} \div \frac{1}{2} =$
$\frac{2}{3} \div \frac{7}{9} =$	$\frac{3}{4} \div \frac{11}{4} =$	$\frac{7}{9} \div 5 =$
$\frac{2}{3} \div \frac{8}{9} =$	$\frac{3}{8} \div \frac{11}{4} =$	$\frac{7}{9} \div \frac{1}{5} =$

Approach II: Testing a Theory

Bill's theory:

"In multiplication or division, you can invert the multiplier or divisor and change the operation sign."

Before we can test that theory, we need to know exactly what he means.

Bill explains. "By 'invert' I mean write the number as a fraction and interchange the numerator and denominator. By 'change the operation sign' I mean change the inverted multiplier to a divisor, or the inverted divisor to a multiplier. According to my theory, the following are true statements, and they illustrate what I mean."

$2 \times \frac{1}{3} = 2 \div \frac{3}{1}$ $\qquad$ $\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1}$

$7\frac{1}{2} \div 2\frac{1}{2} = 7\frac{1}{2} \times \frac{2}{5}$ $\qquad$ $6 \times 4 = 6 \div \frac{1}{4}$

Try your hand with Bill's "invert and change the operation sign" theory.

$8 \times \frac{1}{5} = 8 \div$ $\qquad$ $\frac{2}{3} \div \frac{2}{3} = \frac{2}{3} \times$

$9 \div 9 = 9 \times$ $\qquad$ $\frac{1}{2} \times 2 = \frac{1}{2} \div$

$\frac{3}{4} \div \frac{2}{7} = \_\_\_$ $\qquad$ $\frac{2}{7} \div \frac{3}{4} = \_\_\_$

"You see," Bill bragged, "it works every time!"

"Yes," Anne agreed, "but you made up the examples. Let me make some up. I know that $2 \times 3 = 3 \times 2 = 6$. Let me try your theory."

$2 \times 3 = 2 \div$ and $3 \times 2 = 3 \div$

or $2 \times 3 = 2 \div \frac{1}{3} = 3 \times 2 = 3 \div \frac{1}{2} = 6$

"Is that true?"

Bill smiled. "Try some harder examples."

"All right," Anne agreed.

$$2 \times \frac{1}{2} = \frac{1}{2} \times 2 = 1$$

$$2 \times \frac{1}{2} = 2 \ \boxed{\div \quad} = \frac{1}{2} \times 2 = \frac{1}{2} \ \boxed{\div \quad} = 1$$

$$\frac{2}{3} \times \frac{3}{4} = \frac{2}{3} \ \boxed{\div \quad} = \frac{3}{4} \times \frac{2}{3} = \frac{3}{4} \ \boxed{\quad} = \_\_\_\_$$

$$7 \times 3 = 7 \ \boxed{\quad} = 3 \times 7 = 3 \ \boxed{\quad} = \_\_\_\_$$

Bill was all smiles. "Let me show you some examples I used to check my theory.

"Since $\frac{7}{8} \times \frac{3}{5} = \frac{21}{40}$ and division undoes multiplication, I am sure that

$$\frac{21}{40} \div \frac{3}{5} = \frac{7}{8} \quad \text{and} \quad \frac{21}{40} \div \frac{7}{8} = \frac{3}{5}.$$

"Since $\frac{9}{4} \times \frac{7}{27} = \frac{63}{108} = \frac{7}{12}$, I know that

$$\frac{7}{12} \div \frac{9}{4} = \frac{7}{27} \quad \text{and} \quad \frac{7}{12} \div \frac{7}{27} = \frac{9}{4} = 2\frac{1}{4}.$$

"Try my theory on these examples."

$$\frac{21}{40} \div \frac{3}{5} = \frac{21}{40} \ \boxed{\times \quad} =$$

$$\frac{21}{40} \div \frac{7}{8} = \frac{21}{40} \ \boxed{\quad} =$$

$$\frac{7}{12} \div \frac{9}{4} = \frac{7}{12} \ \boxed{\quad} =$$

$$\frac{7}{12} \div \frac{7}{27} = \frac{7}{12} \ \boxed{\quad} =$$

"I really think my theory will always hold up," Bill concluded.

Anne looked puzzled — almost convinced — but not completely sure. "Too bad it doesn't apply where the fractions are decimal fractions," she sighed.

"Oh, but it will," Bill assured her. "Sometimes it is a real shortcut. For example, I find that dividing by .5 is the same as multiplying by 2:

$$27 \div .5 = 27 \div \frac{5}{10} = 27 \times \frac{10}{5} = 27 \times 2 = 54$$

and multiplying by .5 is the same as dividing by 2:

$$27 \times .5 = 27 \times \frac{5}{10} = 27 \div \frac{10}{5} = 27 \div 2 = \_\_\_\_$$

"Multiplying by .25 is the same as dividing by 4, and dividing by .25 is the same as multiplying by 4:

$$36 \times .25 = 36 \times \frac{25}{100} = 36 \times \frac{1}{4} = 36 \ \boxed{\div \quad} = \_\_\_\_$$

$$36 \div .25 = 36 \div \frac{25}{100} = 36 \div \frac{1}{4} = 36 \ \boxed{\times \quad} = \_\_\_\_$$

"Further, I have less trouble with keeping decimal points in order in examples like this:

$$18.9 \div .07 = 18.9 \div \frac{7}{100} = 18.9 \times \frac{100}{7} = \frac{1890}{7} = \_\_\_\_$$

"I'm exploring lots of examples using my theory and looking for shortcuts. Also, I'm keeping my eyes open for an example in which my theory wouldn't work. I hope I don't find any.

"Here are some of the shortcuts I've found. I use a double-headed arrow ($\longleftrightarrow$) as an abbreviation for *is the same as*."

$\times .25 \longleftrightarrow$	$\div 4$	$\div .2 \longleftrightarrow$	$\times$
$\div \frac{1}{4} \longleftrightarrow$		$\times \frac{1}{8} \longleftrightarrow$	
$\div .05 \longleftrightarrow$		$\times .5 \longleftrightarrow$	
$\div 1\frac{1}{3} \longleftrightarrow$		$\div 1.5 \longleftrightarrow$	
$\div 10\frac{1}{3} \longleftrightarrow$		$\div \frac{2}{3} \longleftrightarrow$	
$\div 1.25 \longleftrightarrow$		$\div 3\frac{1}{3} \longleftrightarrow$	
$\div \frac{3}{17} \longleftrightarrow$		$\div 1\frac{5}{9} \longleftrightarrow$	
$\div 3\frac{1}{7} \longleftrightarrow$		$\div 92\frac{2}{3} \longleftrightarrow$	

Many people refer to part of Bill's theory by saying:

If the divisor is a common fraction, *invert the divisor and multiply.*

Approach III: Undoing Multiplication

Multiplying by a fraction can be considered as "giving instructions to do something."

"How many eggs are there in one-fourth of a dozen?" ______ eggs.

"How many cents are there in one-fourth of a dollar?" ______ cents.

"How many students are there in one-fourth of a class of 28?" ______ students.

Each of these questions asks you to perform an experiment physically or mentally — you are asked to arrange a certain number of things in four groups of the same size and report the number you find in a group.

If we decide to write down the multiplier as the second factor, here are records of the questions:

$12 \times \frac{1}{4} = \_\_\_ \qquad 100 \times \frac{1}{4} = \_\_\_ \qquad 28 \times \frac{1}{4} = \_\_\_$

The computation you performed could also be described equally well by these records:

$12 \div 4 = \_\_\_ \qquad 100 \div 4 = \_\_\_ \qquad 28 \div 4 = \_\_\_$

Suppose that the questions had been: What is three-fourths of a dozen eggs; of 100 cents; of 28 students?

You would probably start out in the same way — to find one-fourth and then multiply by 3. We could record your activity in several ways:

$12 \times \frac{1}{4} \times 3 = \_\_\_ \qquad 100 \times \frac{1}{4} \times 3 = \_\_\_ \qquad 28 \times \frac{1}{4} \times 3 = \_\_\_$

$(12 \div 4) \times 3 = \_\_\_ \qquad (100 \div 4) \times 3 = \_\_\_ \qquad (28 \div 4) \times 3 = \_\_\_$

and the usual form:

$12 \times \frac{3}{4} = \_\_\_ \qquad 100 \times \frac{3}{4} = \_\_\_ \qquad 28 \times \frac{3}{4} = \_\_\_$

If we use N in place of the number of things under consideration, and b as the denominator part of the instructions, and a as the numerator part, we can easily describe this way of multiplying by a fraction by writing the formula:

$$N \times \frac{a}{b} = (N \div b) \times a$$

"How many inches in five-ninths of a yard?" In this case, N is 36, a is 5, and b is 9.

$$N \times \frac{a}{b} = (N \div b) \times a \text{; so, } 36 \times \frac{5}{9} = (36 \div 9) \times 5 = \_\_\_\_$$

In multiplying by a fraction, we carry out two instructions: perform a division and perform a multiplication.

If you analyze your thinking in the previous examples, you probably carried out the division part of the instructions first, and then the multiplication: $(N \div b) \times a$. Suppose that you had reversed the order. Would it have affected the outcome?

$$(N \div b) \times a = (N \times a) \div b$$

Are the following statements true?

$(36 \div 9) \times 5 = (36 \times 5) \div 9 = \_\_\_\_\_$

$(12 \div 4) \times 3 = (12 \times 3) \div 4 = \_\_\_\_\_$

Test other examples. Are you certain that the order doesn't affect the outcome? ______

In fact, you may have been using this idea whenever dividing first would cause complications.

How many days in two-thirds of a week?

$7 \times \frac{2}{3} = (7 \div 3) \times 2 = (7 \times 2) \div 3 = \_\_\_\_$

In the above formula, could N hold a place for a fraction? Does the following formula work?

$$\frac{c}{d} \times \frac{a}{b} = \left(\frac{c}{d} \div b\right) \times a = \left(\frac{c}{d} \times a\right) \div b$$

Try some examples.

How many dozen eggs in two-thirds of a half-dozen?

$\frac{1}{2} \times \frac{2}{3} = \left(\frac{1}{2} \div 3\right) \times 2 = \left(\frac{1}{2} \times 2\right) \div 3 = \_\_\_\_$

How many dozen is half of two-thirds of a dozen eggs?

$\frac{2}{3} \times \frac{1}{2} = \left(\frac{2}{3} \div 2\right) \times 1 = \left(\frac{2}{3} \times 1\right) \div 2 = \_\_\_\_$

$\boxed{\times \frac{a}{b}}$ means $\boxed{\div\, b \text{ and then } \times \text{ by } a}$

or $\boxed{\times\, a \text{ and then } \div \text{ by } b}$

That is, multiplying by a fraction is carrying out a pair of operations — a multiplication (indicated by the numerator) and a division (indicated by the denominator) — and the order does not affect the outcome.

What do we mean by $\div \frac{a}{b}$?

Division is "undoing a multiplication."

If $2 \times 5 = 10$ then $10 \div 5 = 2$.

If $r \times 8 = t$ then $t \div 8 =$ ____.

Suppose that we carry out a pair of operations — a multiplication followed by a division. Then, we use another pair of operations to get back to where we started. Here are some examples.

12	×2	÷3	8	8	÷	×	12
14	×3	÷	6	6	÷	×	14
35	×	÷	14	14	÷	×	35
42	×7	÷6			÷	×	42

same

same

Make up your own examples.

30	×3	÷5			÷	×	30
	×	÷3			÷	×	
	×8	÷			÷	×	
	×	÷9			÷	×	

same

same

In general, for $a \neq 0$ and $b \neq 0$,

$r \times a \div b = s$ ...... $s \div a \times b = r$.

To undo a pair of operations consisting of a multiplication and a division, carry out a division to undo the multiplication and a multiplication to undo the division.

To undo $\times \frac{3}{5}$, you can $\times \frac{5}{3}$

Let us agree that $\div \frac{3}{5}$ is simply shorthand for "undo a multiplication by three-fifths." Then, using a double-headed arrow to mean *produces the same result as,* we can write:

$\div \frac{3}{5} \longleftrightarrow \times \frac{5}{3}$	$\div \frac{4}{1} \longleftrightarrow \times \frac{1}{4}$
$\div \frac{1}{2} \longleftrightarrow \times$ —	$\div \frac{2}{3} \longleftrightarrow \times$ —
$\div \frac{4}{3} \longleftrightarrow \times$	$\div 1\frac{1}{3} \longleftrightarrow \times$
$\div 2\frac{3}{5} \longleftrightarrow \times$	$\div \quad \longleftrightarrow \times 3\frac{2}{3}$

$\boxed{\div \frac{a}{b}}$ means $\div\, a$ and then $\times\, b$

or $\times\, b$ and then $\div\, a$

which can be indicated by $\boxed{\times \frac{b}{a}}$

To undo a multiplication by a and a division by b —

$$n \times \frac{a}{b} = \frac{n \times a}{b}$$

we can divide by a and multiply by b —

$$\frac{n \times \cancel{a}}{\cancel{b}} \times \frac{\cancel{b}}{\cancel{a}} = n$$

Use any method you wish to complete the following statements.

$24 \times \frac{2}{3} =$	$15 \times \frac{3}{5} =$	$12 \times \frac{3}{4} =$
$16 \div \frac{2}{3} =$	$15 \div \frac{3}{5} =$	$9 \div \frac{3}{4} =$
$\frac{2}{5} \div \frac{3}{4} =$	$1\frac{1}{2} \div \frac{1}{5} =$	$\frac{7}{8} \div \frac{2}{3} =$

Approach IV: Mathematical Reasoning

We are going to make some statements and reason from them to a conclusion. The conclusion will give us a procedure for dividing by a fraction.

Here are the statements and some remarks about each one.

(1) For each number a and for each number b (except 0),

$$(a \div b) \times b = a.$$

This statement tells us what is meant by division. If you divide one number by a second and then multiply the answer by the second, the result is the first number.

$$(8 \div 4) \times 4 = 2 \times 4 = 8$$

$(12 \div 3) \times 3 =$ ___ $\times 3 =$ ___

$(3 \div 5) \times 5 =$ ___ $\times 5 =$ ___

Another way of looking at statement (1) is this. If you wish to find out the answer to a division problem (divide a first number by a second), just find out what number should be multiplied by the second number to get the first number.

$30 \div 6 = ?$		$? \times 6 = 30$
1st number, 2nd number		2nd number, 1st number

$40 \div 5 =$ ___	because	___ $\times 5 = 40$
$\frac{1}{4} \div 3 =$ ___	because	___ $\times 3 = \frac{3}{12} = \frac{1}{4}$
$5 \div \frac{1}{2} =$ ___	because	___ $\times \frac{1}{2} = 5$

[Notice that we exclude 0 in statement (1) because it is impossible to divide by 0.]

Now for the second statement.

(2) For each number b (except 0), there is exactly one number — called the *reciprocal* of b — such that

$$b \times \text{the reciprocal of } b = 1.$$

For example, consider the number 3. What number can you multiply 3 by to get 1? Statement (2) says that there is such a number, and that it is called the reciprocal of 3. From what you already know about multiplication, you can see that

$$3 \times \frac{1}{3} = 1.$$

So, $\frac{1}{3}$ is the reciprocal of 3.

Complete each of the following statements.

The reciprocal of 5 is ___ since $5 \times$ ___ $= 1$.

The reciprocal of $\frac{1}{8}$ is ______.

The reciprocal of ______ is .1.

The reciprocal of .1 is ______.

The reciprocal of $\frac{2}{3}$ is ______.

The reciprocal of $\frac{4}{5}$ is ______.

The reciprocal of $\frac{8}{7}$ is ______.

The reciprocal of $\frac{5}{3}$ is ______.

What is the reciprocal of 2? ______

What is the reciprocal of $\frac{1}{3}$? ______

What is the reciprocal of $\frac{3}{4}$? ______

[Do you see why we excluded 0 in statement (2)? Do you think there is a number which is the reciprocal of 0?]

The next two statements are very easy to understand. Check them by testing a few examples.

(3) For all numbers a, b, and c,

$$(a \times b) \times c = a \times (b \times c).$$

Statement (3) tells us that in a multiplication problem you can change the grouping without changing the final answer.

$$(3 \times 5) \times 7 = 15 \times 7 = 105$$
$$3 \times (5 \times 7) = 3 \times 35 = 105$$

Check some examples of statement (3) with fractions and decimals.

(4) For each number a,

$$a \times 1 = a.$$

We doubt that you will need to do any checking to convince yourself of statement (4).

Now let's see what we can get from these four statements.

Consider the division problem:

$$15 \div \tfrac{3}{4} = ?$$

According to statement (1), since $\frac{3}{4}$ is not 0, if we multiply the number $(15 \div \frac{3}{4})$ by $\frac{3}{4}$, we should get 15 as the answer. In other words,

$$(15 \div \tfrac{3}{4}) \times \tfrac{3}{4} = 15.$$

Let's now multiply the numbers on both sides of this equation by the reciprocal of $\frac{3}{4}$. We should get the same answer each time since the numbers are equal.

$$[(15 \div \tfrac{3}{4}) \times \tfrac{3}{4}] \times \begin{pmatrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{pmatrix} = 15 \times \begin{pmatrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{pmatrix}$$

Statement (3) tells us that we can change the grouping in a multiplication problem and not change the answer. Applying this idea to the left side of the last equation, we see that

$$(15 \div \tfrac{3}{4}) \times \left[\tfrac{3}{4} \times \begin{pmatrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{pmatrix}\right] = 15 \times \begin{pmatrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{pmatrix}.$$

Statement (2) tells you that when you multiply a number by its reciprocal, you get 1. So, this equation becomes:

$$(15 \div \tfrac{3}{4}) \times 1 = 15 \times \begin{pmatrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{pmatrix}$$

Statement (4) tells you that when you multiply a number by 1, you get the number you started with. Therefore,

$$15 \div \tfrac{3}{4} = 15 \times \begin{pmatrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{pmatrix}.$$

So, the answer to the original division problem can be found by multiplying 15 by the reciprocal of $\frac{3}{4}$.

From our knowledge of multiplication, we know that $\frac{3}{4} \times \frac{4}{3} = 1$. So, by statement (2), the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$. Thus, we conclude that

$$15 \div \tfrac{3}{4} = 15 \times \tfrac{4}{3}.$$

Dividing 15 by $\frac{3}{4}$ amounts to multiplying 15 by $\frac{4}{3}$. Since $15 \times \frac{4}{3}$ is ______, the answer to the original division problem is ______.

If you look carefully at the preceding discussion you will notice that the particular numbers 15 and $\frac{3}{4}$ did not play an important role in our reasoning until we reached the point where we needed to find the reciprocal of $\frac{3}{4}$. Up to that point we could use the same reasoning starting with *any* numbers (as long as we don't try to divide by 0). This pattern of reasoning is a *proof* of the following statement.

(5) For each number a and for each number b (except 0),

$$a \div b = a \times \text{the reciprocal of } b.$$

The pattern of reasoning shows that statement (5) follows logically from statements (1), (2), (3), and (4).

Statement (5) tells you how to divide by any number (except 0). All you have to do is find the reciprocal of the number and then multiply by it.

For example, suppose that we have the division problem:

$$\tfrac{2}{5} \div \tfrac{7}{8} = ?$$

Statement (5) tells you that (since $\frac{7}{8}$ is not 0) you can find the answer by multiplying $\frac{2}{5}$ by the reciprocal of $\frac{7}{8}$. What is the reciprocal of $\frac{7}{8}$? It must be $\frac{8}{7}$ because $\frac{7}{8} \times \frac{8}{7} = 1$. So,

$$\tfrac{2}{5} \div \tfrac{7}{8} = \tfrac{2}{5} \times \tfrac{8}{7} = \tfrac{16}{35}.$$

Please complete the following statements.

a. $\frac{3}{7} \div \frac{2}{5} = \frac{3}{7} \times \underline{\quad\quad} = \underline{\quad\quad} = 1\frac{1}{14}$

b. $1\frac{1}{3} \div \frac{3}{5} = \frac{4}{3} \times \underline{\quad\quad} = \underline{\quad\quad} = 2\frac{2}{9}$

c. $5\frac{1}{2} \div \frac{2}{7} = \underline{\quad\quad} \times \underline{\quad\quad} = \underline{\quad\quad} = \underline{\quad\quad}$

d. $\frac{5}{8} \div 1\frac{1}{5} = \underline{\quad\quad} \div \frac{6}{5} = \underline{\quad\quad} \times \underline{\quad\quad} = \underline{\quad\quad}$

e. $1\frac{1}{2} \div 1\frac{1}{4} = \underline{\quad\quad} \div \frac{5}{4} = \underline{\quad\quad} \times \underline{\quad\quad} = \underline{\quad\quad} = 1\frac{1}{5}$

f. $9 \div .3 = 9 \div \frac{3}{10} = 9 \times \underline{\quad\quad} = \frac{90}{3} = \underline{\quad\quad}$

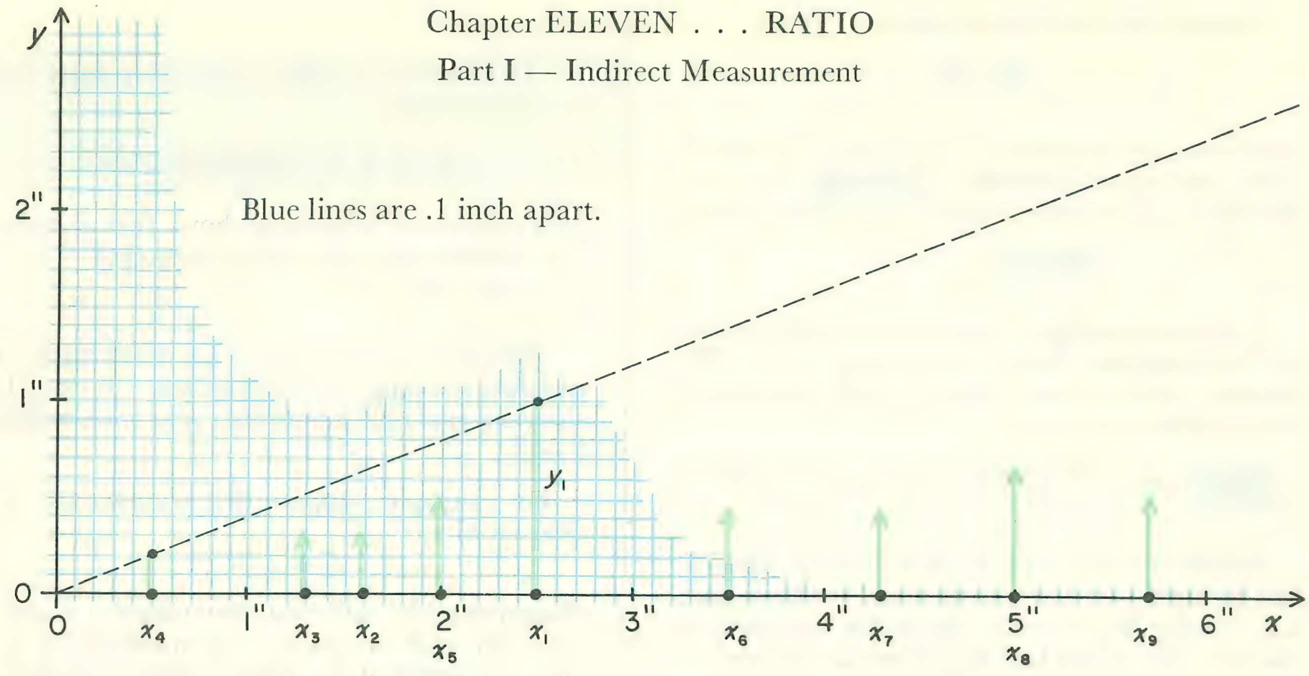
g. $9 \div .03 = 9 \times \underline{\quad\quad} = \underline{\quad\quad} = \underline{\quad\quad}$

h. $2 \div 4 = 2 \times \underline{\quad\quad} = \frac{2}{4} = \underline{\quad\quad}$

You have now seen several approaches to dividing a fraction by a fraction. No matter which approach you use, you will have to do the same computations. These computations are indicated by:

$$\frac{m}{n} \div \frac{p}{q} = \frac{m}{n} \times \frac{q}{p} = \frac{m \times q}{n \times p}$$

Part I — Indirect Measurement



How long (to the nearest tenth of an inch) would each vertical green line be if it were extended from the "*x*-axis" until it meets the broken black line?

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
length parallel to *y*-axis	1"			.2"					
length along the *x*-axis	2.5"			.5"	2"			5"	

Problem: How can you complete the table without drawing a line or making a measurement?

Consider the following cases:

	(1)	(4)	(5)	(8)
y-axis	1"			
x-axis	2.5"	.5"	2"	5"

You can fill in the lengths in examples (4) and (5) by reading the blue lines on the chart.

Think about these entries as part of a *What's My Rule?* game. What is the relationship between the lengths reported in each entry?

$$\frac{y\text{-axis}}{x\text{-axis}} \quad \frac{1}{2.5} = \frac{\quad}{.5} = \frac{\quad}{2} = \frac{\quad}{5}$$

The length parallel to the *y*-axis is two-fifths of the length along the *x*-axis, . . . or it is four-tenths, or forty-hundredths, or any expression in the form of:

$$\frac{2n}{5n}$$

Problem: For all nonzero numbers *n*, *a*, *b*, *c*, is it the case that

$$\frac{2n}{5n} = \frac{2a}{5a} = \frac{2b}{5b} = \frac{2c}{5c} = \frac{2}{5}\ ?$$

Before answering for sure, convince yourself that common fractions, decimal fractions, and positive and negative numbers would be suitable as replacements for *n, a, b,* and *c*.

Problem: How could the idea we are considering here be used to find the height of a flag pole?

Problem: How could the idea be used to measure the length of a flag pole that was bent over at its base so that it was no longer straight up and down?

While you are thinking about those problems, let's look at a relationship between any two pairs of entries in our table: (We'll omit the ″ mark).

(1)	(4)
1	2
2.5	.5

(5)	(8)
2	5

(1)	(5)
1	
2.5	2

Multiply opposite corner entries as indicated by the blue lines.

1 x .5 = ____   ____ x 5 = ____

2.5 x .2 = ____   2 x ____ = ____

1 x 2 = ____

2.5 x ____ = ____

Since (8) has measurements involving only whole numbers, let's use that fact and the idea above to find other entries for the table.

(8)	(2)
2	
5	1.6

(8)	(3)
2	
5	1.3

(8)	(6)
2	
5	3.5

(8)	(7)
2	
5	4.3

2 x 1.6 = 3.2   2 x 1.3 = 2.6

5 x ____ = 3.2   5 x ____ = 2.6

2 x 3.5 = ____   2 x 4.3 = ____

5 x ____ = ____   5 x ____ = ____

Distances along the x-axis are indicated in the table below. How long would the green lines have to be to reach up to the broken diagonal line? (All numbers are numbers of inches.)

	(10)	(11)	(12)	(13)	(14)	(15)
y-axis						
x-axis	3	7	9	.3	70	.09

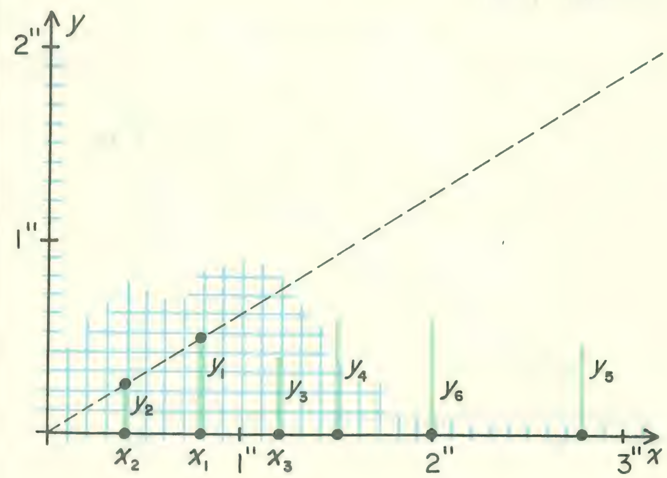
How far out along the x-axis would you need to go so that the vertical distance to the broken line is the number of inches indicated?

	(16)	(17)	(18)	(19)	(20)	(21)
y-axis	5	11	14	.05	1.1	1400
x-axis						

We hope that you found a shortcut for these examples.

Problem: Does the relationship we have been considering depend on our starting with a 2-to-5 "ratio"? Suppose that we had drawn a graph that had a 5-to-8 ratio — 5 up for every 8 over.

Let's draw such a graph and study it.



Make entries to the nearest hundredth inch.

	(1)	(2)	(3)	(4)	(5)
y-axis					
x-axis	.8	.4			

	(6)	(7)	(8)	(9)	(10)	(11)
y						
x		8	10	18	2.4	54

(7)	(5)
5	
8	2.8

(7)	(8)
5	
8	10

(7)	(9)
5	
8	18

(7)	(11)
5	
8	54

5 x 2.8 = ____   5 x 10 = ____

8 x ____ = ____   8 x ____ = ____

5 x 18 = ____   5 x 54 = ____

8 x ____ = ____   8 x ____ = ____

For all numbers a, b, c, and d ($b \neq 0$ and $d \neq 0$),

if $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$

and

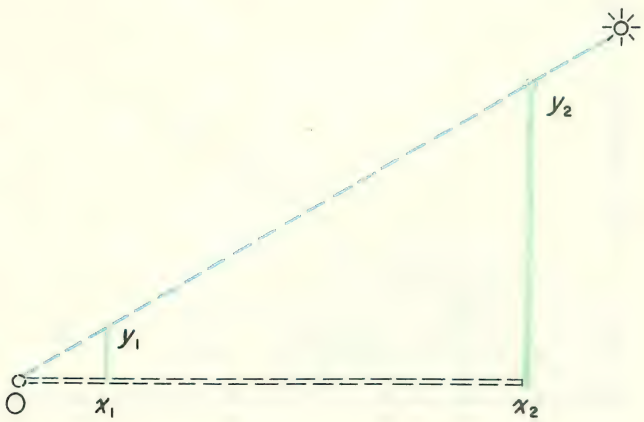
if $ad = bc$ then $\frac{a}{b} = \frac{c}{d}$.

Do you believe that statement? ______

About the height of those flag poles . . .

You have a 5-foot rod and a tape measure.

One method makes use of the shadow of the flag pole. The 5-foot rod is placed on the shadow so that the shadow of its tip and the shadow of the tip of the flag pole are at the same point.



We know that y_1 is 5′. We measure the distance from O (tip of the shadows) to x_1 (bottom of the rod) and find that it is $8\frac{1}{2}$′; and we measure the distance from O to x_2 (bottom of the flag pole) and find it is 51′.

Now we record our information:

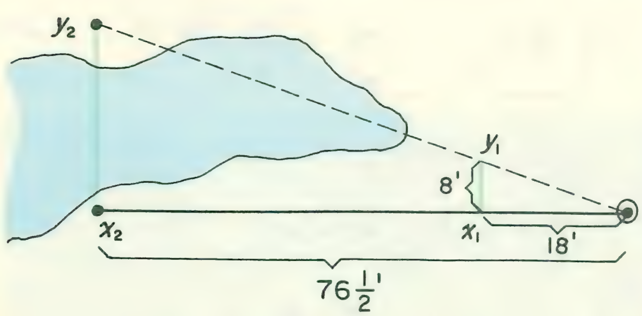
	(1)	(2)
y-axis	5′	
x-axis	$8\frac{1}{2}$′	51′

So, the height of the flag pole is ______ ft.

Can you think of a method you could use if the sun weren't out?

Could you use this method to find the height of your room, a building, a tree?

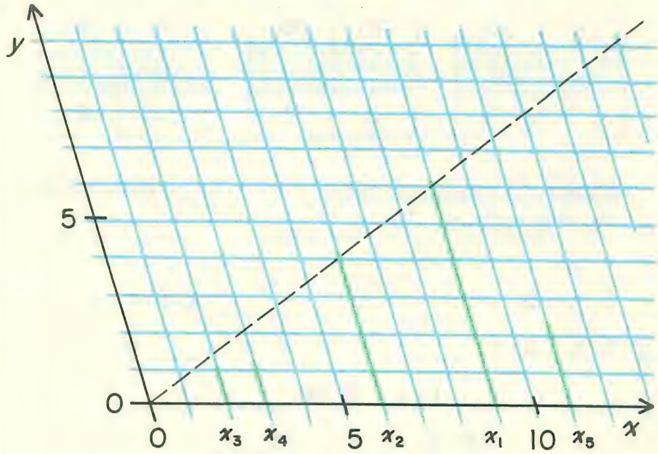
Can you find the distance between two points with a lake between them?



	(1)	(2)
y-axis		
x-axis		

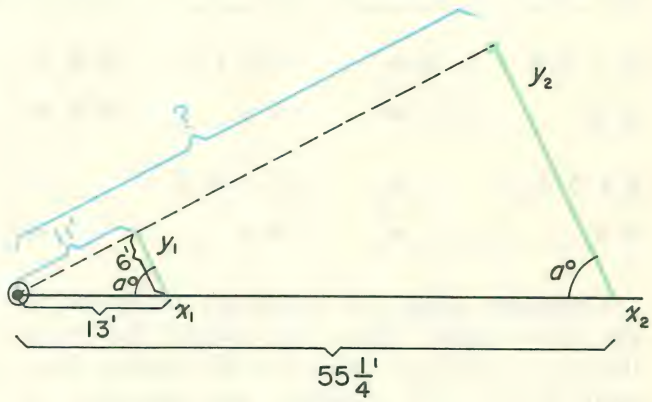
. . . about the bent flag pole . . .

Let's consider a grid in which the x-axis and y-axis are not perpendicular to each other.



	(1)	(2)	(3)	(4)	(5)
y-axis					
x-axis	9				

Do your results suggest a way to go about measuring the length of the bent flag pole?



	(1)	(2)
y-axis		
x-axis		

The length of the flag pole is ______ ft.

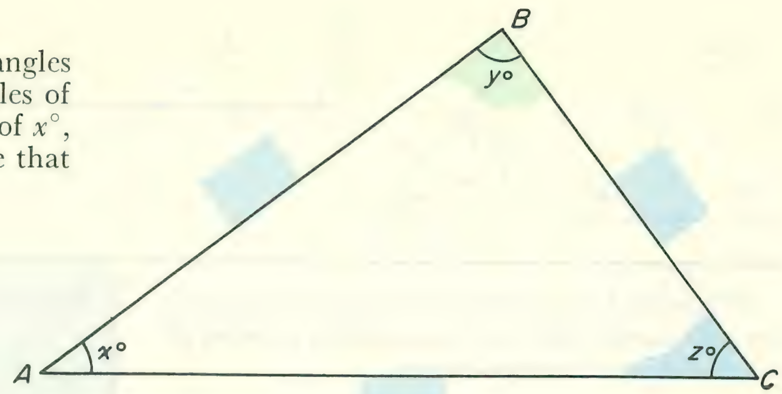
(You would need to take care that the angle of the 6′ rod was the same as that of the flag pole and, sighting from point O, you would need to be sure that the bottoms and tips of the rod and pole were in line.)

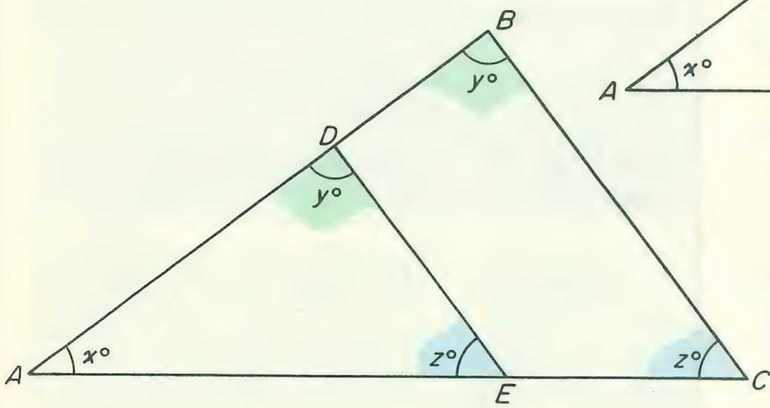
Problem: If the point of sight (O) is 11′ from the tip of the 6′ rod, how far is the point of sight from the tip of the flag pole?

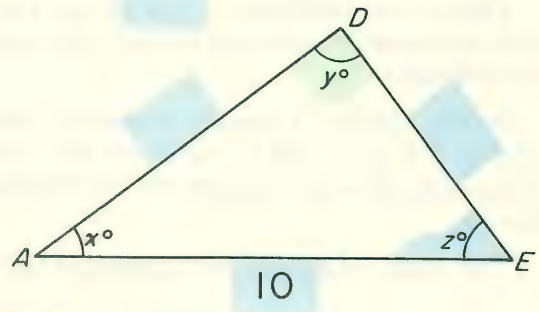
Answer: ______ ft.

I.

Two triangles are *similar* if the three angles of one are equal in size to the three angles of the other. That is, a triangle with angles of x°, y°, and z° is similar to any other triangle that has angles of x°, y°, and z°.







In the blue blocks above, record the number of quarter-inches in each side. Use a ruler to find the lengths.

Use the blue blocks below to check the following statements:

II.

$$\frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE} \qquad \frac{AB}{BC} = \frac{AD}{DE} \qquad \frac{BC}{AC} = \frac{DE}{AE} \qquad \frac{AB}{AC} = \frac{AD}{AE}$$

$$\frac{12}{8} = \frac{\quad}{\quad} = \frac{\quad}{\quad} \qquad \frac{12}{\quad} = \frac{\quad}{\quad} \qquad \frac{\quad}{\quad} = \frac{\quad}{\quad} \qquad \frac{\quad}{\quad} = \frac{\quad}{\quad}$$

III.

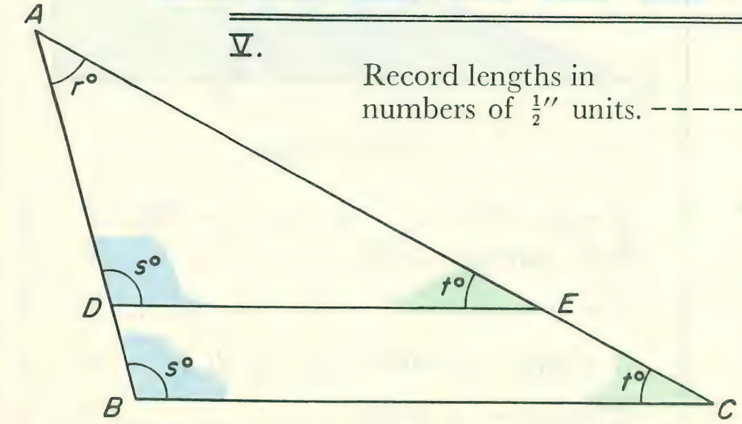
$$AB \times DE = AD \times BC \qquad BC \times AE = DE \times AC \qquad AB \times AE = AD \times AC$$

$$\square \times \square = \square \times \square \qquad \square \times \square = \square \times \square \qquad \square \times \square = \square \times \square$$

IV.

$$\frac{AB - AD}{AC - AE} = \frac{AB + AD}{AC + AE} = \frac{AB}{AC} = \frac{AD}{AE} \qquad \frac{BC - DE}{AC - AE} = \frac{BC + DE}{AC + AE} = \frac{BC}{AC} = \frac{DE}{AE}$$

$$\frac{\square - \square}{\square - \square} = \frac{\square + \square}{\square + \square} = \frac{\quad}{\quad} = \frac{\quad}{\quad} \qquad \frac{\square - \square}{\square - \square} = \frac{\square + \square}{\square + \square} = \frac{\quad}{\quad} = \frac{\quad}{\quad}$$



V.

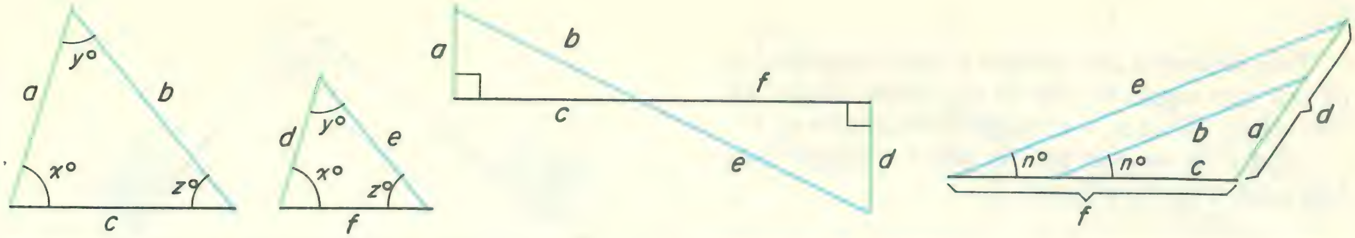
Record lengths in numbers of $\frac{1}{2}''$ units. ------→

AB	AC	BC	AD	AE	DE

Are all the statements made above also true about the triangles at the left? Try a few:

$$\frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE} \qquad \frac{AB}{AC} = \frac{AD}{AE}$$

$$\frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad} \qquad \frac{\quad}{\quad} = \frac{\quad}{\quad}$$



When the lengths of sides of similar triangles are compared, there are a surprising number of relationships we can point out.

These relationships do not depend on the unit of measure selected nor on the shape of the triangles.

In the similar triangles sketched above, a and d, b and e, and c and f are the numbers of units of measure in pairs of "corresponding" sides.

The basic relationship is expressed by:

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

This is often read as "a is to d as b is to e as c is to f." It is sometimes written:

$$a:d = b:e = c:f$$

We can make several statements about relationships among the numbers $a, b, c, d, e,$ and f.

Do you believe that

$$\text{if } \frac{a}{d} = \frac{b}{e} \text{ then } a \times e = d \times b?$$

Do you believe that

if $a \times e = d \times b$ or $ae = db$

$$\text{then } \frac{a}{d} = \frac{b}{e}, \quad \frac{d}{a} = \frac{e}{b},$$

$$\frac{a}{b} = \frac{d}{e}, \quad \frac{b}{a} = \frac{e}{d}$$

?

$3 \times 21 = 7 \times 9$ . . . is it true that

$$\frac{3}{7} = \frac{9}{21}, \quad \frac{7}{3} = \frac{21}{9},$$

$$\frac{3}{9} = \frac{7}{21}, \quad \frac{9}{3} = \frac{21}{7}$$

?

Suppose that

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}.$$

Which of the statements in the green box below follow from this?

$$ae = db \qquad bf = ec \qquad af = dc$$

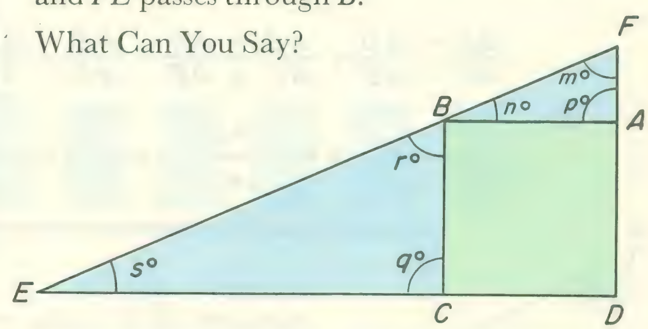
$$\frac{a}{d} = \frac{c}{f} \qquad \frac{b}{c} = \frac{e}{f} \qquad \frac{f}{d} = \frac{c}{a}$$

$$\frac{b-a}{e-d} = \frac{c}{f} \qquad \frac{b+c}{e+f} = \frac{a}{d} \qquad \frac{a+c}{d+f} = \frac{b-c}{e-f}$$

Make up statements expressing relationships among the lengths of sides of the pairs of similar triangles shown above.

A bit of review: $ABCD$ is a square, AF is an extension of DA, CE is an extension of DC, and FE passes through B.

What Can You Say?



True or false:

$q = p = 90$. . T F $s + r = 180$. . T F

BA is parallel to ED. . . . . T F

$s = n$. . . . . T F $r = m$. . . . . T F

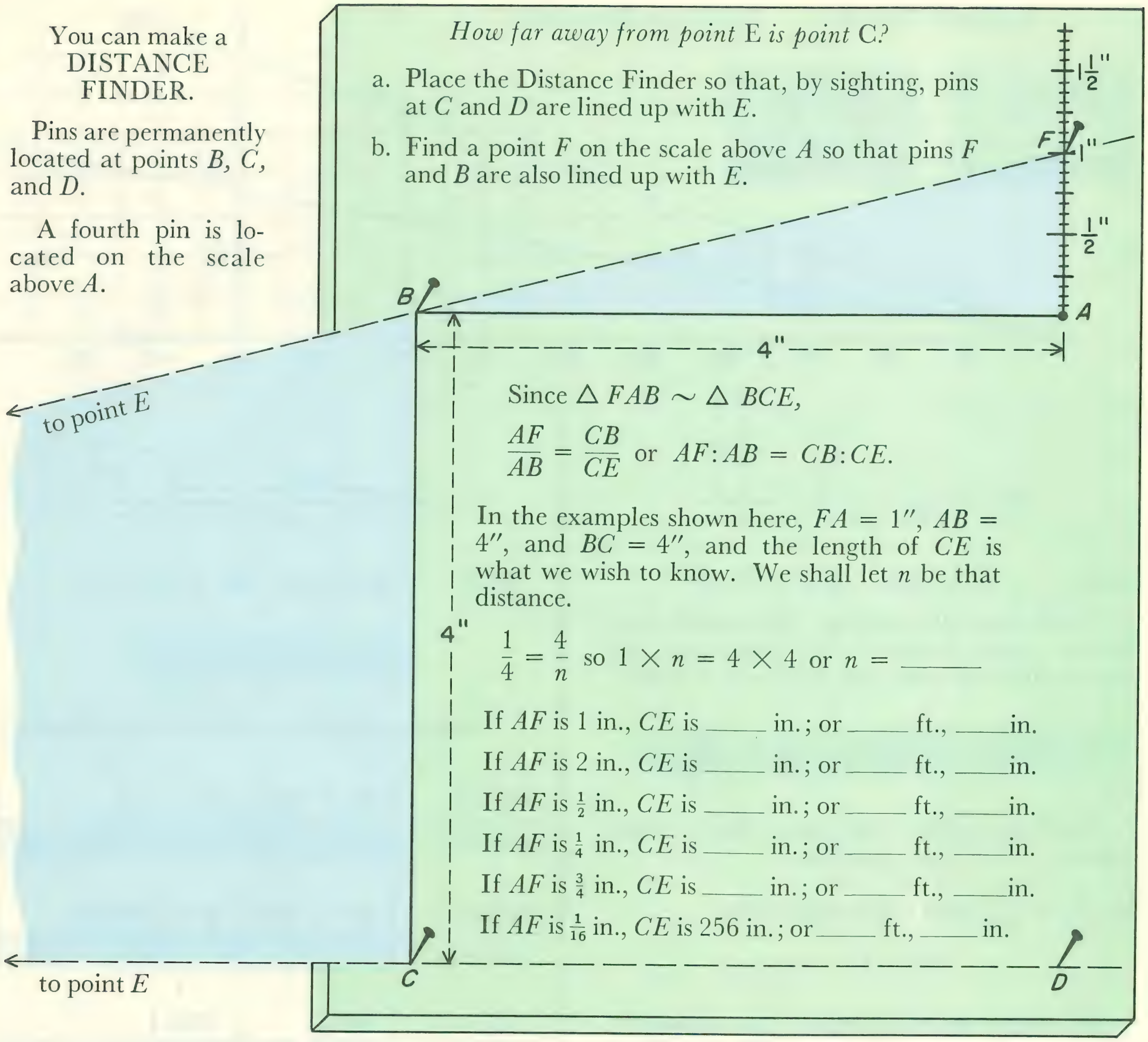
$\triangle FAB \sim$ (is similar to) $\triangle BCE$. . . . T F

$\triangle BAF \sim \triangle EDF$ . . . . T F

You can make a DISTANCE FINDER.

Pins are permanently located at points B, C, and D.

A fourth pin is located on the scale above A.



How far away from point E *is point* C*?*

a. Place the Distance Finder so that, by sighting, pins at C and D are lined up with E.

b. Find a point F on the scale above A so that pins F and B are also lined up with E.

Since $\triangle FAB \sim \triangle BCE$,

$$\frac{AF}{AB} = \frac{CB}{CE} \text{ or } AF:AB = CB:CE.$$

In the examples shown here, $FA = 1''$, $AB = 4''$, and $BC = 4''$, and the length of CE is what we wish to know. We shall let n be that distance.

$$\frac{1}{4} = \frac{4}{n} \text{ so } 1 \times n = 4 \times 4 \text{ or } n = \_\_\_\_\_$$

If AF is 1 in., CE is _____ in.; or _____ ft., _____ in.

If AF is 2 in., CE is _____ in.; or _____ ft., _____ in.

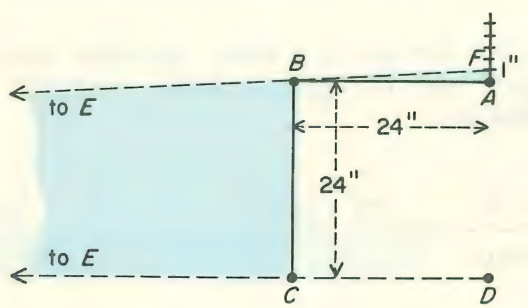
If AF is $\frac{1}{2}$ in., CE is _____ in.; or _____ ft., _____ in.

If AF is $\frac{1}{4}$ in., CE is _____ in.; or _____ ft., _____ in.

If AF is $\frac{3}{4}$ in., CE is _____ in.; or _____ ft., _____ in.

If AF is $\frac{1}{16}$ in., CE is 256 in.; or _____ ft., _____ in.

For greater accuracy and longer distances, consider a model in which AB and BC are 2 feet. Now, when AF is 1″, we have this situation:



AF is 1″, AB is 24″, CB is 24″, CE is n''

$$\frac{1}{24} = \frac{24}{n}, n = 24 \times 24 = \_\_\_\_\_ \text{ inches}$$

or _____ ft., _____ in.

Here is a table for use with the Distance Finder for longer distances described on the left. *(Please complete the table.)*

AF	CE ft.	CE in.
$1''$	48	0
$\frac{15}{16}''$		
$\frac{7}{8}''$		
$\frac{13}{16}''$		
$\frac{3}{4}''$		
$\frac{11}{16}''$		
$\frac{5}{8}''$		
$\frac{9}{16}''$		

AF	CE ft.	CE in.
$\frac{1}{2}''$		0
$\frac{7}{16}''$		
$\frac{3}{8}''$		
$\frac{5}{16}''$		
$\frac{1}{4}''$		
$\frac{3}{16}''$		
$\frac{1}{8}''$		
$\frac{1}{16}''$		

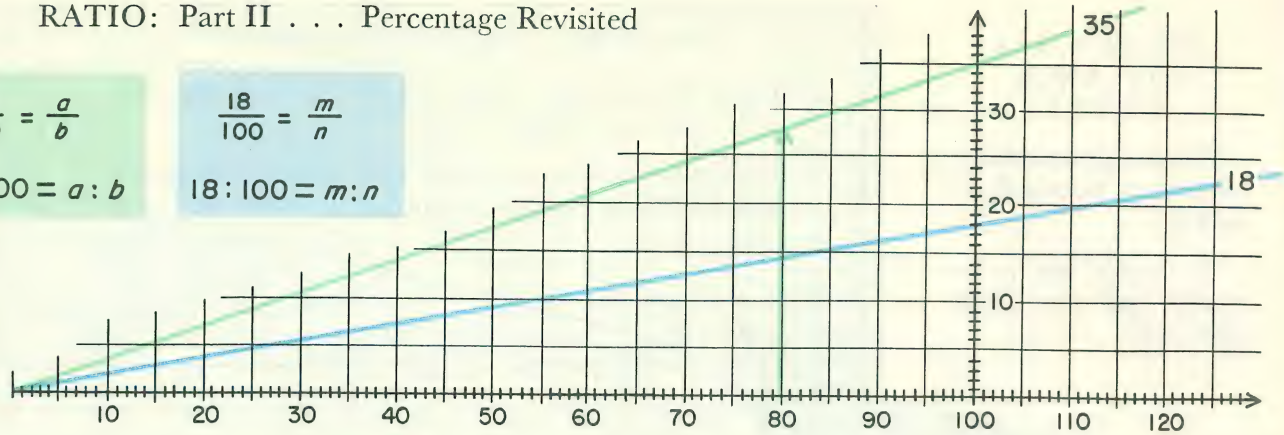
RATIO: Part II . . . Percentage Revisited

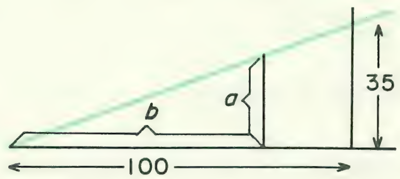
$\frac{35}{100} = \frac{a}{b}$

$35:100 = a:b$

$\frac{18}{100} = \frac{m}{n}$

$18:100 = m:n$





35% *. . . at the same rate as 35 to 100*

There were 80 problems. Jim missed them at the rate of 35 out of 100. How many (a) out of 80 is the same rate as 35 out of 100?

$a:80 = 35:100$ or $\frac{a}{80} = \frac{35}{100}$

Read the graph. Jim must have missed about ________ problems.

If $\frac{a}{80} = \frac{35}{100}$ then $100a = 80 \times 35$

or $a =$ ______.

In another set of examples, Jane missed 21. This is at the rate of 35 out of 100 . . . 35%. How many problems (a) were there in the set?

$21:a = 35:100$ or $\frac{21}{a} = \frac{35}{100}$

If $\frac{21}{a} = \frac{35}{100}$ then $35a = 21 \times$ ______

or $a =$ ______.

Use the graph to complete the following:

35% of 55 is about ______.

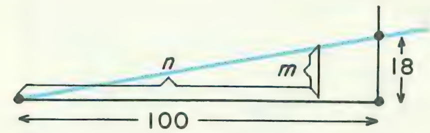
35% of 42 is about ______.

35% of 105 is about ______.

35% of ______ is about 15.

35% of ______ is about 30.

35% of ______ is about 9.



18% *. . . at the same rate as 18 to 100*

$18:100 = m:n$

Read the graph to complete the following:

18% of . . .	35	70	110			
is about . . .				10	$22\frac{1}{2}$	18

18% of . . .	67	672	6.68	6694	
is about . . .					24

18% of . . .	83		830		
is about . . .		1.5		300	3

18% of . . .	112		450	22.5	
is about . . .		40			.2

With the use of a ruler (or other straight-edge) and the graph above, complete the following:

26% of . . .	75	120				265
is about . . .			51	17	26	

15% of . . .	67			33		113
is about . . .		19	15		12	

Did you notice shortcuts to help you check your results?

I. "We pay 4% interest per annum (each year) on all savings accounts."

$\frac{4}{100} = \frac{\text{interest}}{\text{savings}}$, $.04 \times \text{savings} = \text{interest}$

Interest at 4%

Amount	1 year	2 years	8 months
$15.00	$.60	$1.20	$.40
$100.00			
$115.00			
	$1.20		
	$1.80		
		$32.00	
$515.00			
$560.00			
	$.30		
		$1.80	
			$.30
$18.75			
			$1.00

II. 15% off all regular prices

$\frac{15}{100} = \frac{\text{reduction}}{\text{reg. price}}$, $.15 \times \text{reg. price} = \text{reduction}$

Regular Price	Reduction	Sale price
$100.00		
$80.00		
$20.00		
$8.00		
$92.00		
$192.00		
	$14.40	
		$782.00
	$2.25	
$45.00		
		$76.50
	$20.25	
	$2.70	
		$2.21
		$12.92

III. The won-lost records of baseball teams compare games won with games played. The percent is figured to the closest tenth of a percent, and written as thousandths.

$\frac{n}{100}$ or $\frac{10n}{1000} = \frac{\text{games won}}{\text{games played}}$

In 1965, each National League baseball team played 162 games in the regular season (except Philadelphia and St. Louis; each played only 161 games).

Complete the season's records given below in part.

	Games Won	Lost	Rating
Los Angeles	97	65	
San Francisco	95		
Pittsburgh		72	
Cincinnati			.549
Milwaukee		76	
Philadelphia	85	76	
St. Louis			.497
Chicago			.444
Houston	65		
New York			.309

IV. The National Geographic Society reports that about 71%, or 139,434,000 square miles, of the earth's total surface area is water. This is about 61% of the Northern Hemisphere and about 81% of the Southern Hemisphere.

$$\frac{71}{100} = \frac{139{,}434{,}000 \text{ square miles}}{\text{total surface area of the earth}}$$

The total surface area of the earth is about ______________ square miles. In the Northern Hemisphere, there are about ______________ square miles of land; in the Southern Hemisphere, there are about ______________ square miles of land.

The total land surface area of the earth is about ______________ square miles.

Over 7 and up 1, for $\frac{1}{7}$, does not lead to as much accuracy as over 70 and up 10. And over 132 and up 180, for $\frac{180}{132}$, takes us off the chart; but over 66 and up 90 can be handled.

ratio	I look up	approx. equiv. to
$\frac{1}{7}$	$\frac{10}{70}$	$\frac{14}{100}$
$\frac{180}{132}$	$\frac{90}{66}$	$\frac{136}{100}$
$\frac{4}{9}$	$\frac{40}{\quad}$	$\frac{\quad}{100}$
$\frac{123}{112}$	$\frac{61\frac{1}{2}}{\quad}$	$\frac{\quad}{100}$
$\frac{3}{11}$	$\frac{\quad}{88}$	$\frac{\quad}{100}$
$\frac{150}{110}$	$\frac{75}{\quad}$	$\frac{\quad}{100}$
$\frac{7}{8}$	$\frac{\quad}{80}$	$\frac{\quad}{100}$
$\frac{15}{8}$	$\frac{\quad}{80}$	$\frac{\quad}{100}$
$\frac{1}{6}$		$\frac{\quad}{100}$
$\frac{5}{6}$		$\frac{\quad}{100}$
$\frac{7}{6}$		$\frac{\quad}{100}$
$\frac{5}{8}$		$\frac{\quad}{100}$
$\frac{9}{7}$		$\frac{\quad}{100}$
$\frac{7}{13}$		$\frac{\quad}{100}$
$\frac{220}{178}$		$\frac{\quad}{100}$
$\frac{3}{57}$		$\frac{\quad}{100}$

ratio	I look up	approx. equiv. to
$\frac{3}{8}$		$\frac{\quad}{100}$
$\frac{1\frac{3}{8}}{\quad}$		$\frac{\quad}{100}$
$\frac{7}{19}$		$\frac{\quad}{100}$
$\frac{26}{19}$		$\frac{\quad}{100}$
$\frac{64}{360}$		$\frac{\quad}{100}$
$\frac{1\frac{1}{2}}{21}$		$\frac{\quad}{100}$
$\frac{2\frac{2}{3}}{30}$		$\frac{\quad}{100}$
$\frac{1}{200}$		$\frac{\quad}{100}$
$\frac{.5}{200}$		$\frac{\quad}{100}$

$$\frac{.5}{200} = \frac{n}{100}$$

$$200 \times n = .5 \times 100$$

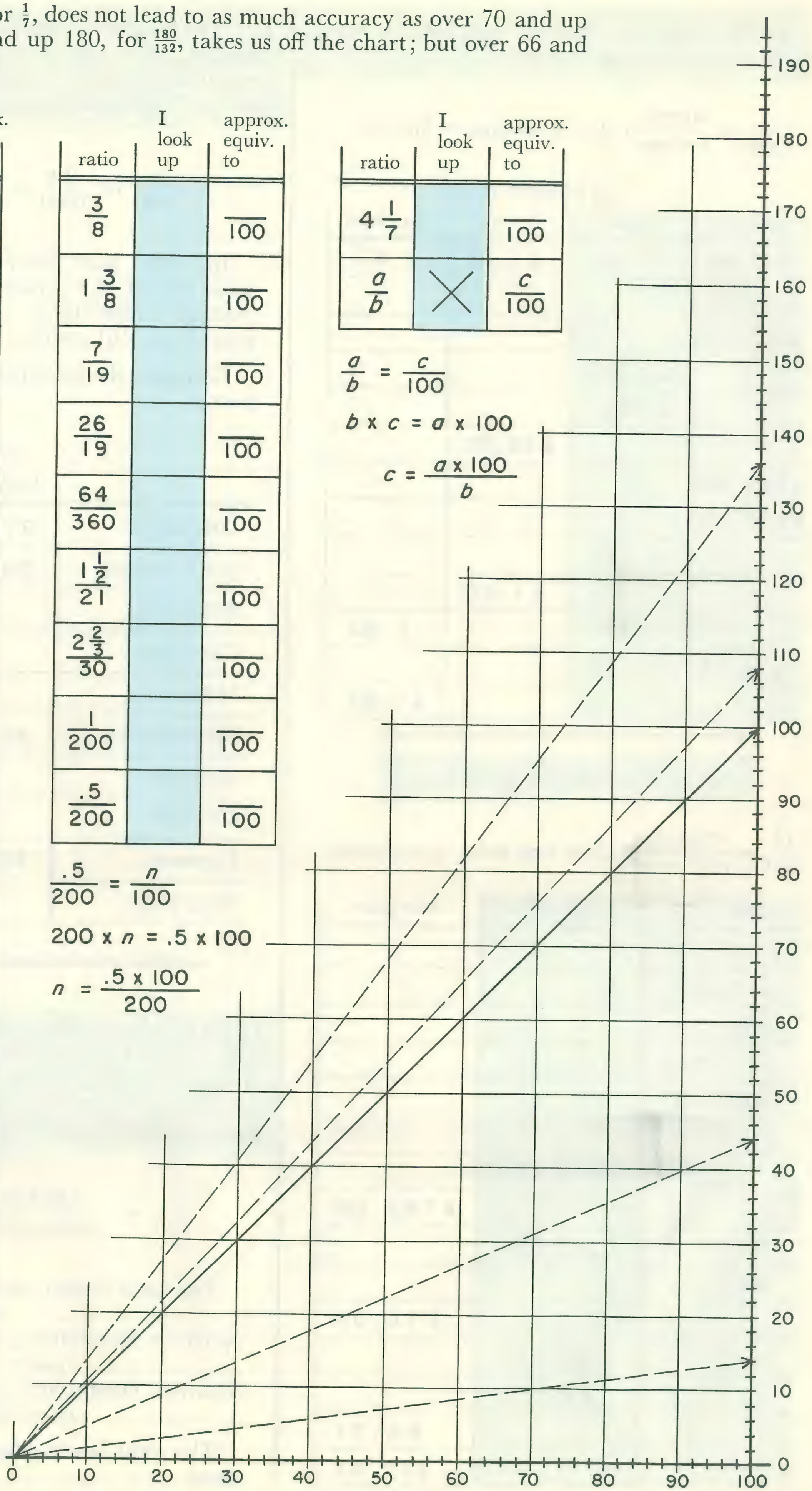
$$n = \frac{.5 \times 100}{200}$$

ratio	I look up	approx. equiv. to
$4\frac{1}{7}$		$\frac{\quad}{100}$
$\frac{a}{b}$	X	$\frac{c}{100}$

$$\frac{a}{b} = \frac{c}{100}$$

$$b \times c = a \times 100$$

$$c = \frac{a \times 100}{b}$$



A. Mr. Baxter received an annual interest payment of \$25.50 on an investment of \$850. What was the annual rate of interest?

$\frac{n}{100} = \frac{\text{interest}}{\text{investment}}$ or $n:100 = \text{int.}:\text{inv.}$

The annual rate of interest is ______%.

B. Forty-five students are absent. That is 5% of the enrollment. What is the enrollment?

$\frac{5}{100} = \frac{\text{absentees}}{\text{enrollment}}$ $5:100 = \text{abs.}:\text{enr.}$

The total enrollment is ______ students.

C. Mrs. Allen earns a commission of 12%. In a certain period, she reported \$1,525.00 sales. What is her commission for that period?

$\frac{\quad}{100} = \frac{\text{commission}}{\text{sales}}$

Mrs. Allen is entitled to \$________ in commission.

D. Mr. Martin borrowed some money for a year. He paid \$22 interest at the rate of 4%. How much did he borrow?

$\frac{\quad}{100} = \text{\_\_\_\_\_\_}$

He must have borrowed \$__________.

E. Barry gave correct answers to 34 out of 40 problems. What percent did he get correct, and what percent did he miss?

$\frac{n}{100} = \text{\_\_\_\_\_\_}$

Barry had ______% of the answers correct. He missed ______% of the questions.

F. 120 is 30% of what?

$\frac{30}{100} = \text{\_\_\_\_\_\_}$

120 is 30% of ________.

G. 150% of 700 is ________.

$\frac{\quad}{100} = \text{\_\_\_\_\_\_}$

H. 12 is ______% of 16.

______ = ______

I. 16 is ______% of 12.

______ = ______

J. $\frac{\quad}{100} = \frac{28}{35}$

K. $\frac{\quad}{100} = \frac{91}{650}$

L. $\frac{7}{100} = \frac{\quad}{420}$

M. $\frac{65}{100} = \frac{\quad}{352}$

N. $\frac{19}{100} = \frac{9.5}{\quad}$

O. $\frac{31}{100} = \frac{155}{\quad}$

P. $\frac{125}{100} = \frac{35}{\quad}$

Q. $\frac{200}{100} = \frac{62.5}{\quad}$

R. $\frac{\quad}{100} = \frac{437.50}{250}$

S. $\frac{\quad}{100} = \frac{88.14}{78}$

T. $\frac{180}{100} = \frac{\quad}{29}$

U. $\frac{300}{100} = \frac{\quad}{17}$

V. $\frac{5}{100} = \frac{\quad}{350}$

W. $\frac{2.5}{100} = \frac{\quad}{350}$

X. $\frac{19}{100} = \frac{\quad}{500}$

Y. $\frac{9\frac{1}{2}}{100} = \frac{\quad}{500}$

Z. $\frac{25}{100} = \frac{\quad}{125}$

a. $\frac{\quad}{100} = \frac{31.25}{250}$

b. $\frac{\quad}{100} = \frac{982.8}{756}$

c. $\frac{\quad}{100} = \frac{491.4}{378}$

d. 50 % is ______% of 200 %

RATIO: Part III . . . Reporting Remainders

$$\begin{array}{r} 3\frac{1}{5} \\ ?\overline{)\;?\;} \end{array}$$

22% are absent

When we select a particular way to express a ratio, we often lose a little bit of history.

Consider the way we report results of an example involving computation.

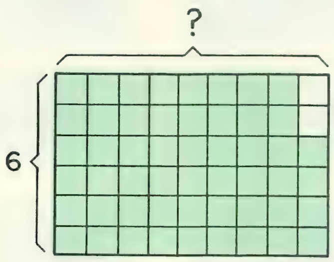
What was the division problem that has $3\frac{1}{5}$ as its quotient?

How many students are absent if 22% are absent?

Remainders

It is usually considered good form to report a remainder as a fraction in its "lowest terms."

How many columns of tile will there be if we try to arrange 52 tiles, 6 to a column.



Consider three ways of reporting results:

(a) 8 columns and 4 tiles left over.

(b) 8 and $\frac{2}{3}$ columns.

(c) 8 full columns and 4 out of 6 in another column.

Which way of reporting gives the most information?

Now consider these three reports of results of another grouping of tiles.

(a) 4 columns and 2 tiles left over.

(b) 4 and $\frac{1}{4}$ columns.

Neither report by itself provides enough information to know just what the activity was.

However, consider this report:

(c) 4 full columns and 2 out of 8 in another column.

Now you know that there were ______ tiles to be arranged, ______ to a column.

The two usual ways of reporting remainders are:

$$\begin{array}{r} 4\text{ R}3 \\ 12\overline{)\,51} \end{array} \quad \text{and} \quad \begin{array}{r} 4\frac{1}{4} \\ 12\overline{)\,51} \end{array}$$

If this were a "tile" example, the ratio of 3 out of 12 for the remainder column is changed to its "simplest form" — and consequently a bit of its history is lost.

To preserve this history, we might agree to make a full report:

$$51 \div 12 = 4\frac{3}{12}$$

Or, we might decide to give both of the other reports for a division example. If you think about it, you can reconstruct the following examples:

a. $\begin{array}{r} 12\text{ R}4 \\ \square\overline{)\;\square\;} \end{array}$ and $\begin{array}{r} 12\frac{1}{2} \\ \square\overline{)\;\square\;} \end{array}$

b. $\begin{array}{r} 8\text{ R}6 \\ \square\overline{)\;\square\;} \end{array}$ and $\begin{array}{r} 8\frac{2}{3} \\ \square\overline{)\;\square\;} \end{array}$

c. $\begin{array}{r} 24\text{ R}4 \\ \square\overline{)\;\square\;} \end{array}$ and $\begin{array}{r} 24\frac{4}{7} \\ \square\overline{)\;\square\;} \end{array}$

d. $\begin{array}{r} 13\text{ R}9 \\ \square\overline{)\;\square\;} \end{array}$ and $\begin{array}{r} 13\frac{3}{4} \\ \square\overline{)\;\square\;} \end{array}$

In the following, the remainder describes the full history. Can you reconstruct the examples?

e. $\begin{array}{r} 14\frac{3}{9} \\ \square\overline{)\;\square\;} \end{array}$

g. $\begin{array}{r} 24\frac{15}{25} \\ \square\overline{)\;\square\;} \end{array}$

f. $\begin{array}{r} 17\frac{7}{14} \\ \square\overline{)\;\square\;} \end{array}$

h. $\begin{array}{r} 15\frac{12}{15} \\ \square\overline{)\;\square\;} \end{array}$

22% are absent

In a similar way, reports such as the one above do not include their history.

How many students are there if all are present? We might be reporting that

11 out of ______ are absent;

22 out of ______ are absent;

99 out of ______ are absent;

154 out of ______ are absent;

385 out of ______ are absent.

Thus, a report that "55 out of 250 are absent" tells us more than the report that "22% are absent."

In another school, the report was:

114 out of 475 are absent

That is a full report . . . but which school has the higher rate of absence?

(a) 55 out of 250, or

(b) 114 out of 475

These are the full reports, but they are hard to compare. We know that school (a) has a rate of 22 out of 100. What is the rate at school (b)?

(a) 55:250 = ______:100

(b) 114:475 = ______:100

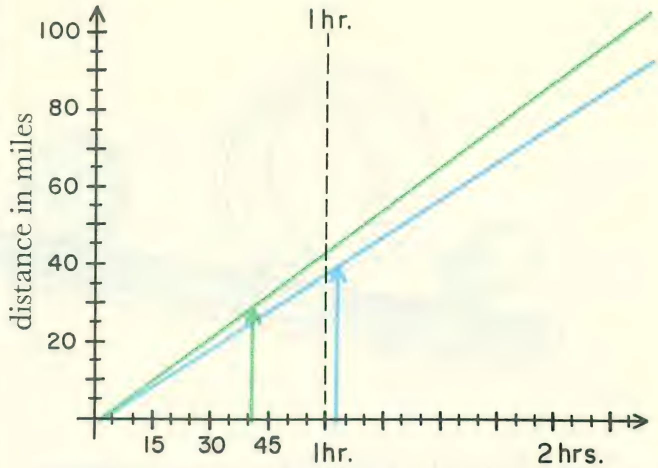
If we preserve the full history in our reports, it may be hard to compare them with others.

If we report in terms of "per cents" then comparisons will usually be easier.

Team A has won 9 out of 20 games and team B has won 11 out of 25 games. Which has the better record?

Answer: ______, because ______________.

Consider a graph showing distances traveled in specified times.



time

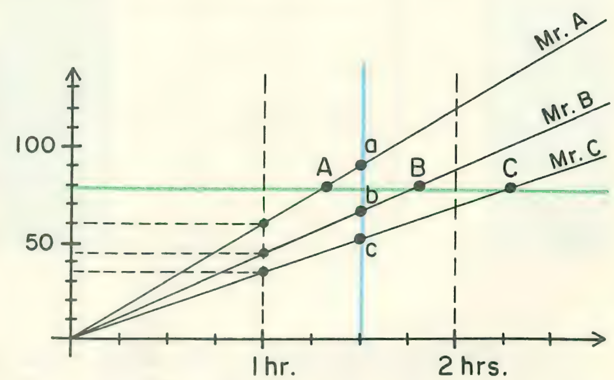
Mr. Newton traveled 30 miles in 40 minutes, and Mr. Walsh traveled 40 miles in 1 hour and 2 minutes.

The lines in the graph show the rates at which the two are traveling. We can tell that Mr. Newton is traveling at about ______ miles per hour, and Mr. Walsh at about ______ miles per hour.

By looking at the vertical arrows, we can tell the exact nature of the trip each reported — the time he had to travel the number of miles given.

But here is another report.

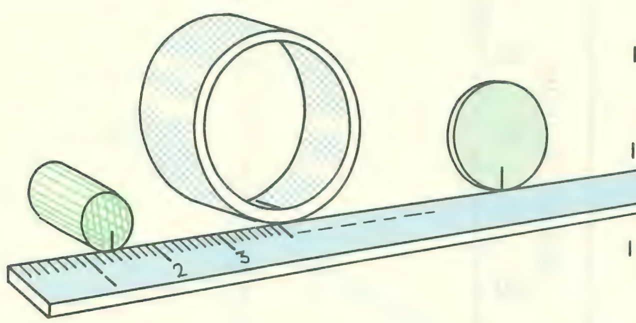
Mr. Albert was traveling at 60 miles per hour, Mr. Bailey at 45 miles per hour, and Mr. Collins at 35 miles per hour.



Now we know nothing but their rates of travel . . . but we have lots of information available.

1. What do points *A, B,* and *C* on the green line suggest?

2. What do points *a, b,* and *c* on the blue line suggest?

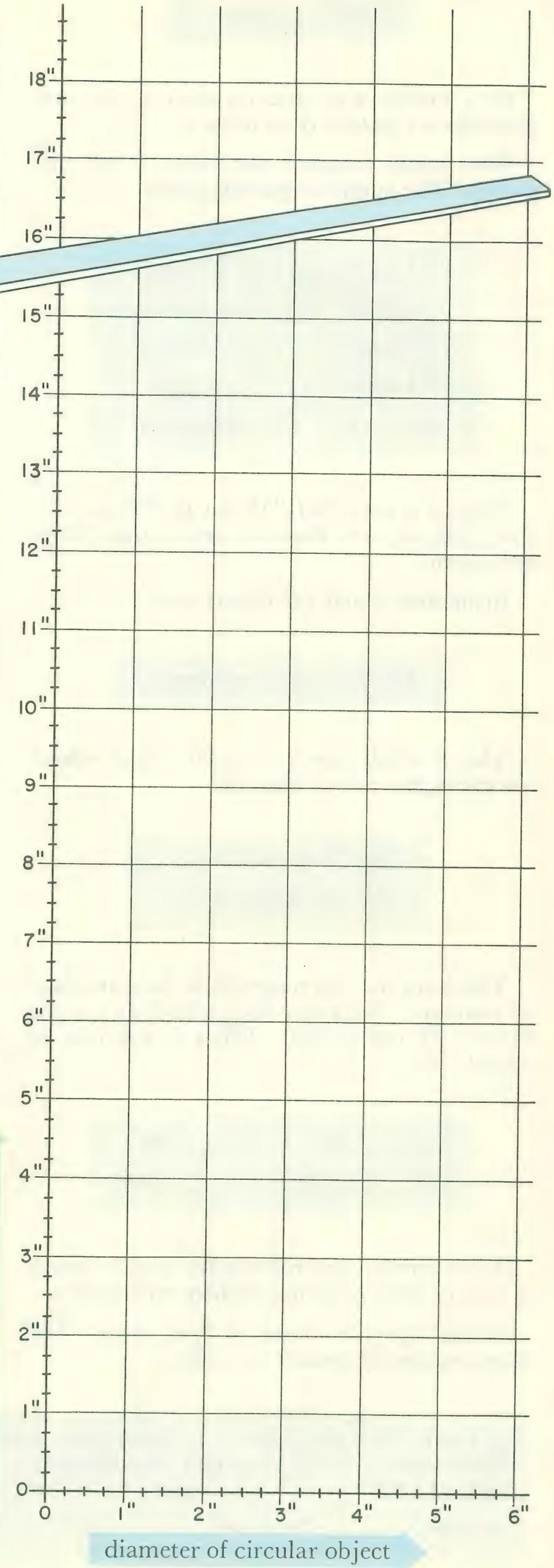


You will need a yardstick or a tape measure.

Out of cardboard, cut wheels with diameters of 1″, 2″, 3″, 3½″, 4″, 5″, and 6″. Near their outside edges, put a mark so you can tell when they have moved through one turn. Roll them along the yardstick. Record in the table below and the graph on the right the distances each rolls in four experiments. Add the results and divide by four to find "average distance."

All measurements are in inches.

diam.	distance in 1 turn				avg.
	1st	2nd	3rd	4th	
1					
2					
3					
$3\frac{1}{2}$					
4					
5					
6					



Continue experimenting with other objects such as tin cans, rolls of tape, wheels — anything that is circular. Record as many of the results as possible in the graph.

Describe what you notice in the graph.

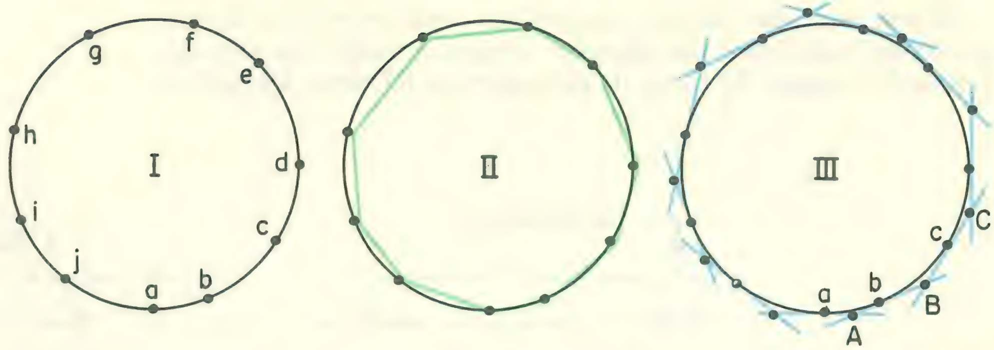
____

____

____

Another Experiment.

Draw a circle any size. Mark some points on the circle, any distance apart — they need not be equally spaced (see I).



Connect these points with line segments (see II).

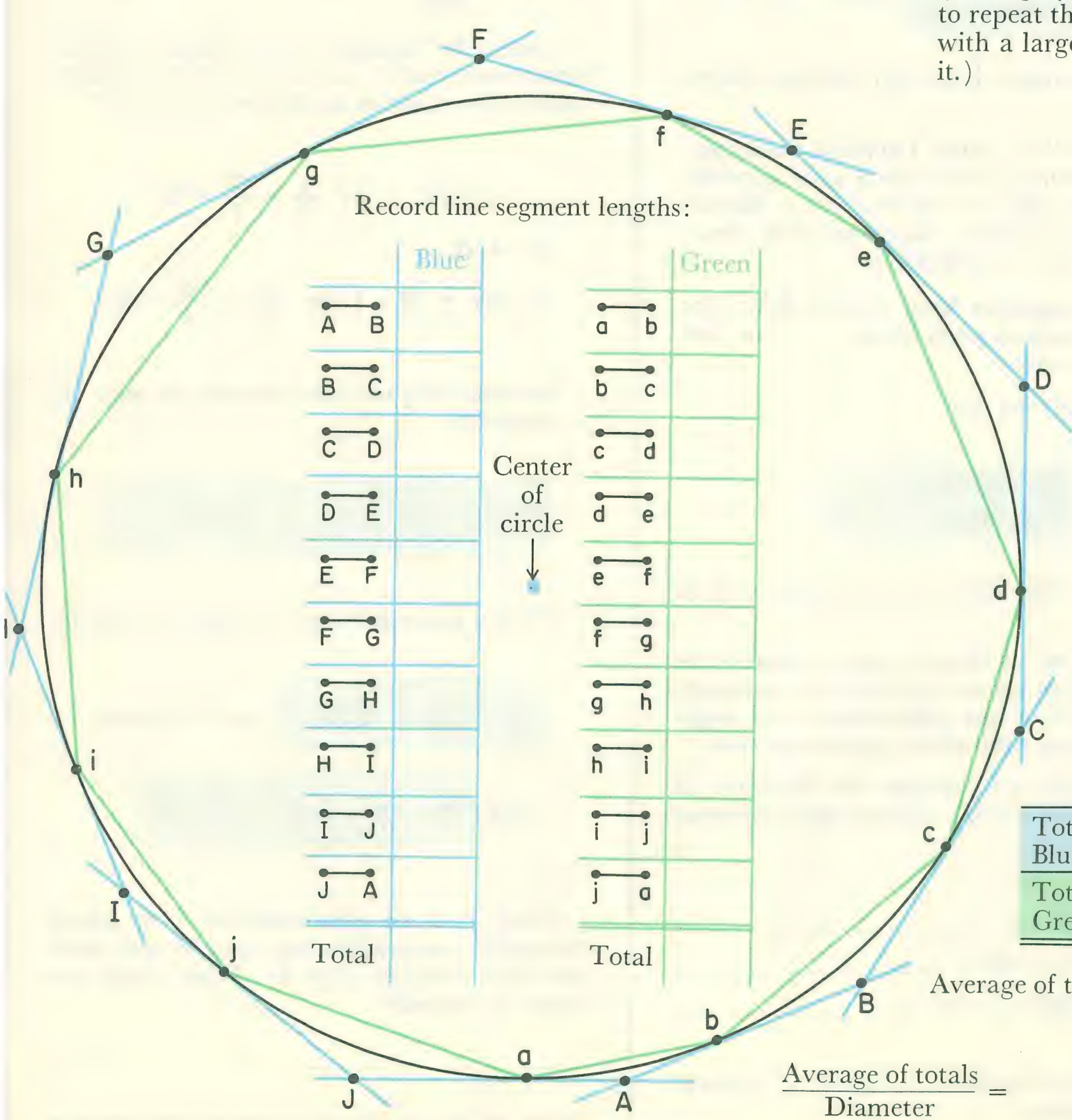
Draw line segments outside the circle that touch it only at the points you have already indicated. Where they cross each other, locate more points (see III).

(Below is a single, enlarged drawing that we shall use.)

Measure each green line segment and add the results. Using the second group of points, measure each blue line segment and add the results.

Why must the distance around the circle — called its *circumference* — be somewhere between these two totals?

(Perhaps you would like to repeat this experiment with a larger circle. Try it.)



Record line segment lengths:

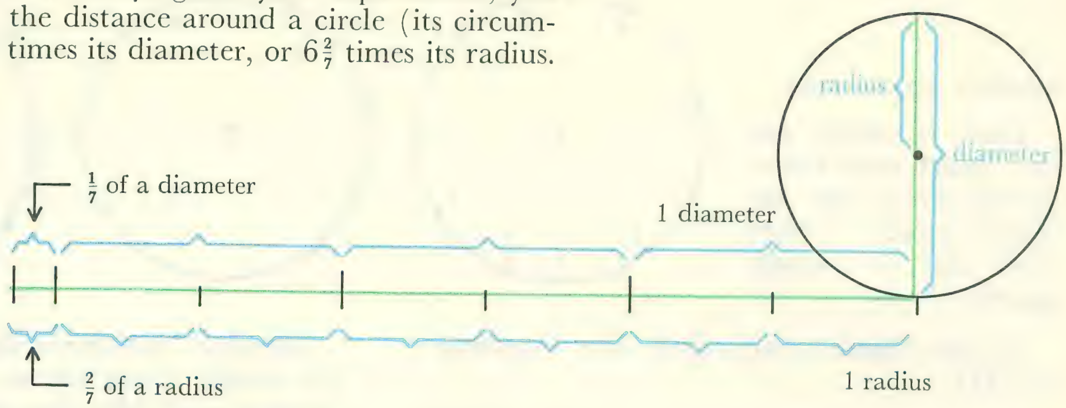
	Blue
A B	
B C	
C D	
D E	
E F	
F G	
G H	
H I	
I J	
J A	
Total	

	Green
a b	
b c	
c d	
d e	
e f	
f g	
g h	
h i	
i j	
j a	
Total	

Total of Blue	
Total of Green	
Average of totals	

$$\frac{\text{Average of totals}}{\text{Diameter}} =$$

If you were careful in carrying out your experiments, you probably found that the distance around a circle (its circumference) is about $3\frac{1}{7}$ times its diameter, or $6\frac{2}{7}$ times its radius.



$3.14159265358979323846264338327950288419716939937510 \approx \pi$

The ratio $\dfrac{\text{circumference}}{\text{diameter}}$ cannot be exactly expressed, no matter how many decimal places we use.

About 300 B.C., some Egyptian mathematicians found that 3.1416 was a close approximation to this ratio. In about 1579, a Roman by the name of Vieta, found an even closer approximation: 3.1415926535.

Modern computers have written down the first 10,000 decimal place digits . . . the first 50 are given above.

So, we simply say that

$$\frac{\text{circumference}}{\text{diameter}} = \pi.$$

π is a Greek letter. We pronounce it as "pie."

Then, we use as close an approximation as we need. Often we use $3\frac{1}{7}$; for more accuracy, we use 3.1416 — but astronomers and engineers sometimes need closer approximations.

Suppose that we consider the diameter of the earth as 8,000 miles and use three different approximations:

(a) $3\frac{1}{7} \times 8{,}000 =$ ____________

(b) $3.1416 \times 8{,}000 =$ ____________

(c) $3.14159265 \times 8{,}000 =$ ____________

The difference between (a) and (b) is about ________ miles.

The difference between (b) and (c) is about ________ feet.

Using d for diameter, r for radius, c for circumference, and π as the ratio of the circumference of a circle to its diameter, we write:

$$c : d = \pi : 1 \quad \text{or} \quad \frac{c}{d} = \frac{\pi}{1} = \pi$$

$$2r = d$$

$$c : 2r = \pi : 1 \quad \text{or} \quad \frac{c}{2r} = \frac{\pi}{1} = \pi$$

Remembering our investigation on page 78, we can write:

$$c = d \times \text{\_\_\_\_} \qquad c = 2r \times \text{\_\_\_\_}$$

The last two statements are usually given as:

$$c = \pi d = 2\pi r$$ and it follows

that $d = \dfrac{c}{\pi}$ and $r = \dfrac{c}{\text{\_\_\_}}$.

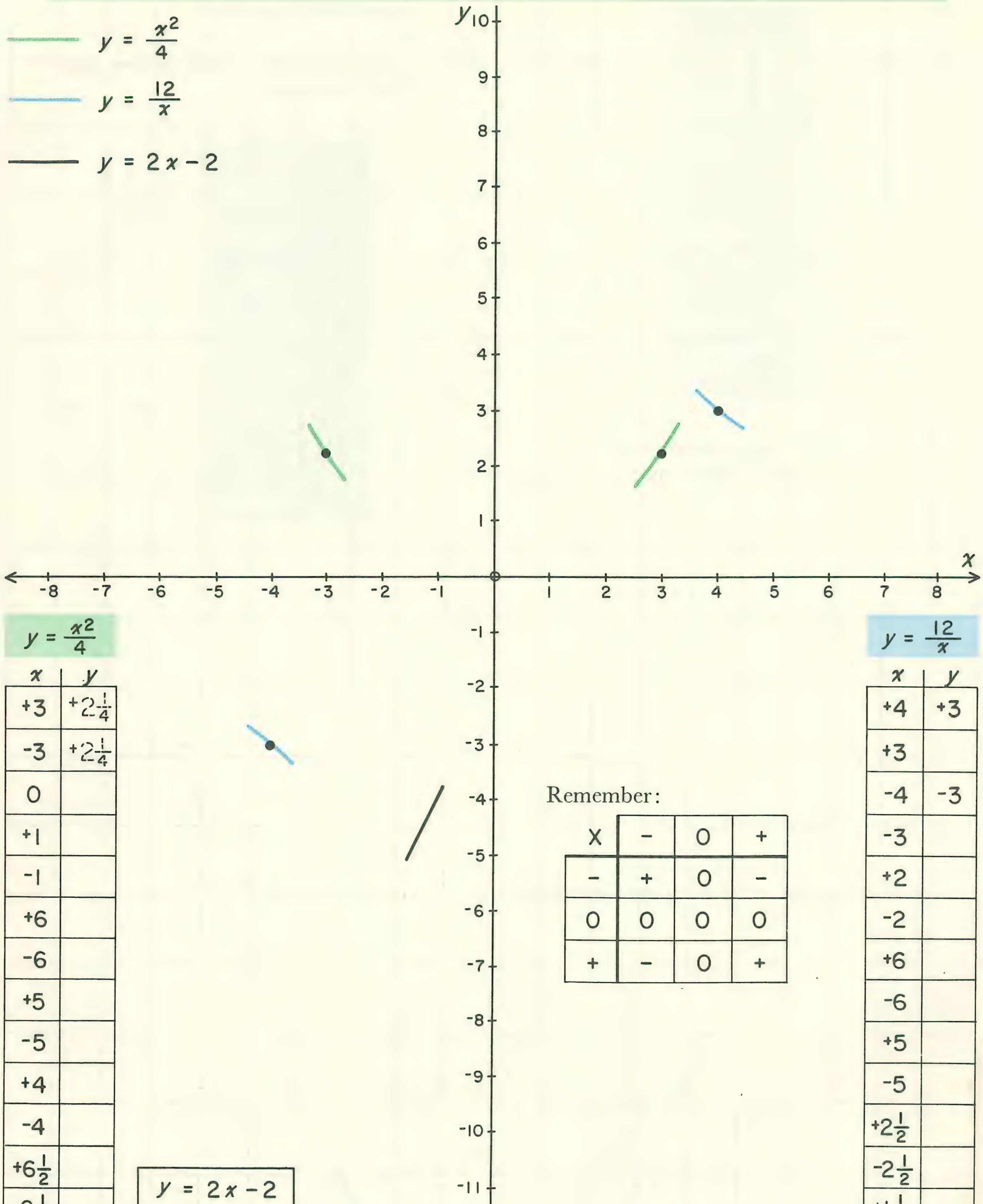
Using $3\frac{1}{7}$ as an approximation to π, check the results you graphed on page 86 and check your final result on page 87. What would you report in general?

____

____

Chapter TWELVE . . .

RELATIONSHIPS THAT MAY NOT LEAD TO STRAIGHT LINE GRAPHS



$y = \frac{x^2}{4}$

x	y
+3	$+2\frac{1}{4}$
-3	$+2\frac{1}{4}$
0	
+1	
-1	
+6	
-6	
+5	
-5	
+4	
-4	
$+6\frac{1}{2}$	
$-2\frac{1}{2}$	
$+\frac{1}{2}$	
$+5\frac{1}{2}$	

$y = \frac{12}{x}$

x	y
+4	+3
+3	
-4	-3
-3	
+2	
-2	
+6	
-6	
+5	
-5	
$+2\frac{1}{2}$	
$-2\frac{1}{2}$	
$+1\frac{1}{5}$	
$-1\frac{1}{5}$	
$+1\frac{5}{7}$	

Remember:

X	−	0	+
−	+	0	−
0	0	0	0
+	−	0	+

$y = 2x - 2$

x	+5	-3	+1	+4	+3	-1	-4	+6	$-1\frac{1}{2}$	0	+2	-2
y	+8	-8										

$y = x^2$

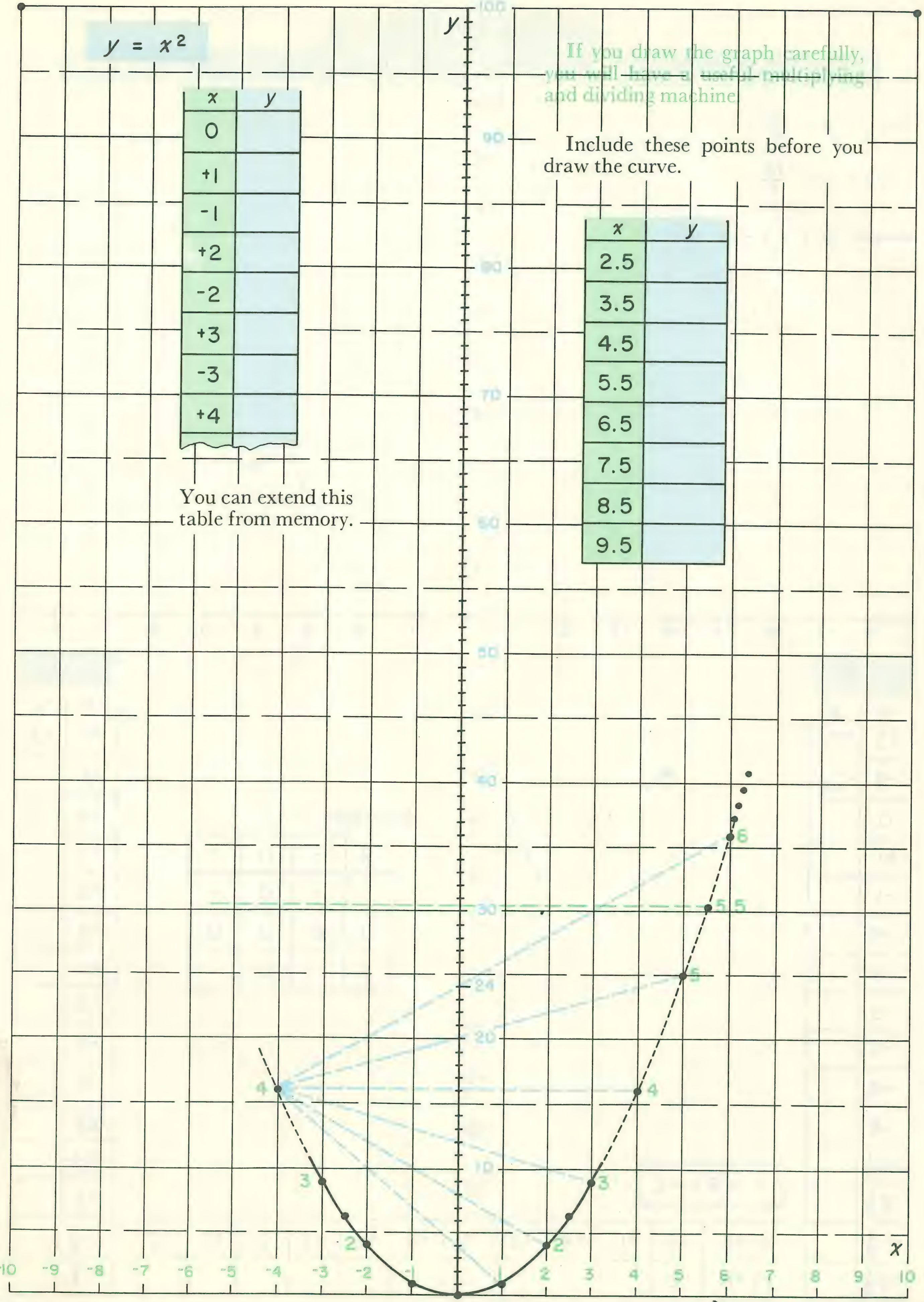
x	y
0	
+1	
-1	
+2	
-2	
+3	
-3	
+4	

You can extend this table from memory.

If you draw the graph carefully, you will have a useful multiplying and dividing machine.

Include these points before you draw the curve.

x	y
2.5	
3.5	
4.5	
5.5	
6.5	
7.5	
8.5	
9.5	



Perhaps you remember a shortcut for squaring any 2-digit number that ends in 5. If not, these examples and hints may save lots of time with 2.5^2, 3.5^2, etc.

I.

$$\begin{array}{r} 2.5 \\ \times\ 2.5 \\ \hline 6.25 \end{array} \qquad \begin{array}{r} 3.5 \\ \times\ 3.5 \\ \hline 12.25 \end{array} \qquad \begin{array}{r} 4.5 \\ \times\ 4.5 \\ \hline 20.25 \end{array} \qquad \begin{array}{r} 5.5 \\ \times\ 5.5 \\ \hline \end{array}$$

Use your graph on page 90 to approximate the following:

II.

	approx.
6 x 3.5	
7.5 x 8	
5 x 6.5	
8.5 x 5	

	approx.
9.5 x 7	
8 x 6.5	
5 x 8.5	
7.5 x 9	

Were your approximations close? If you notice the right-hand digits of the factors, you can be sure about the right-hand digit in the product. With this help, you can turn your approximations into exact answers.

How can you use the graph to give answers for the following?

III.

8 x 45	
55 x 7	
4 x 65	

75 x 7	
9 x 65	
45 x 5	

Suppose that we wish to improve our graph by locating more points on both halves of the curve. Let's begin by locating points whose x-coordinates are 6.1, 6.2, 6.3, and 6.4.

IV.

6.1 x 6.1 = ________
6.2 x 6.2 = ________
6.3 x 6.3 = ________
6.4 x 6.4 = ________

Perhaps you can develop some team work to mark off both halves of the curve in tenths.

If so, you have a graph that will help give you very close approximations for the following:

V.

6.2 x 7.5	
9 x 63	
45 x 64	
6.1 x 85	
.55 x 63	

8.1 x 79	
57 x .48	
$68\frac{1}{2} \times 73\frac{1}{2}$	
$8.9 \times 95\frac{1}{2}$	
8.6 x 8.6	

As you have undoubtedly noticed, the graph is as useful in division as it is in multiplication.

Find the approximate quotients (or missing factors) in the following by using the graph:

VI.

70 ÷ 7.5	
43 ÷ 6.1	
49 ÷ 8	
$35\frac{3}{4} \div 5.5$	
$6\frac{3}{8} \div 8\frac{1}{2}$	

56.7 ÷ 6.3	
24.75 ÷ .45	
$518\frac{1}{2} \div 85$	
8645 ÷ 91	
59.5 ÷ .85	

Using the graph, find approximate answers to the following:

VII.

	approx.
72^2	
4.4^2	
$.83^2$	
6.5^2	

	approx.
53^2	
9.001^2	
5.9^2	
28^2	

VIII. What Are My Rules?

$\sqrt{36} = 6$ $\quad\sqrt{81} = 9$ $\quad\sqrt{.16} = .4$

$\sqrt{100} =$ $\quad\sqrt{.09} =$ $\quad\sqrt{25} =$

$\sqrt{400} =$ $\quad\sqrt{64} =$ $\quad\sqrt{1.44} =$

Using the graph, find approximate answers for the following:

IX.

	approx.
$\sqrt{50}$	7.1
$\sqrt{90}$	
$\sqrt{40}$	

	approx.
$\sqrt{59}$	
$\sqrt{.45}$	
$\sqrt{.75}$	

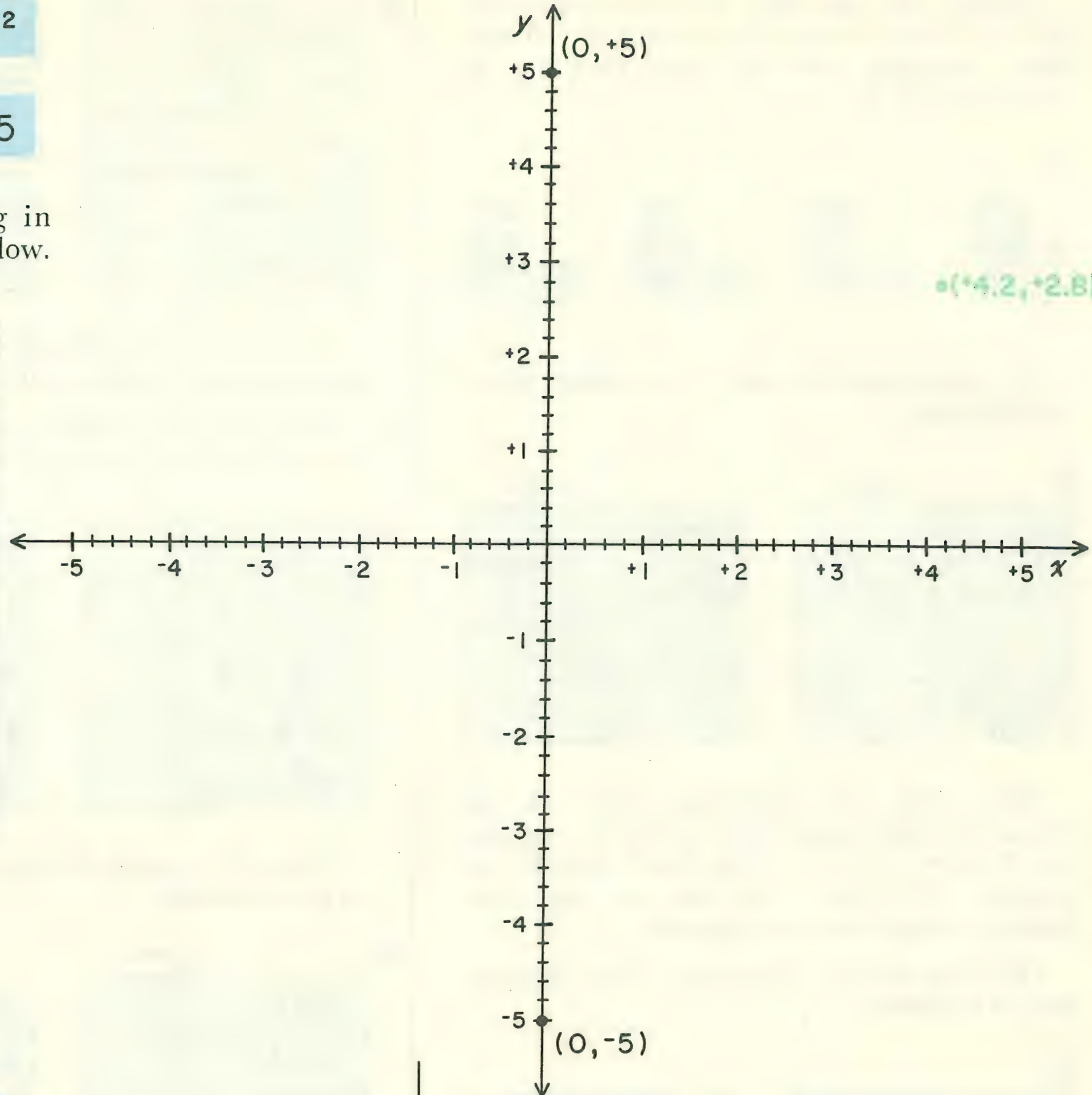
$$y^2 = 25 - x^2$$

or

$$y^2 + x^2 = 25$$

Before filling in the table, read below.

x	y
0	$^+5$, $^-5$
+3	
-3	
-4	
+4	
+5	
-5	
+2	$^+\sqrt{21}$, $^-\sqrt{21}$
-2	



If $y^2 + x^2 = 25$ and $x = 0$ then $y^2 + 0 = 25$; so, $y^2 = 25$. But, $^+5 \times {}^+5 = 25$ and $^-5 \times {}^-5 = 25$. So, if $y^2 = 25$ then $y = {}^+5$ or $y = {}^-5$. Hence, when $x = 0$, $y = {}^+5$ or $y = {}^-5$. This gives us two points to graph: $(0, {}^+5)$ and $(0, {}^-5)$

When $x = {}^+3$, $x^2 = 9$. Then $y^2 + 9 = 25$. So, $y^2 = 16$ and $y = {}^+$______ or $y = {}^-$______. This gives us two points: $({}^+3,\ )$ and $({}^+3,\ )$

When $x = {}^-3$, $x^2 = 9$. Again, $y^2 + 9 = 25$. So, $y^2 = 16$ and $y =$ ______ or $y =$ ______. We have two more points: $({}^-3,\ )$ and $(\ ,\ )$

When $x = {}^-4$, $x^2 =$ ______. So, $y^2 +$ ______ $= 25$. Hence, $y^2 =$ ______ and $y =$ ______ or $y =$ ______. Two more points: $({}^-4,\ )$ and $(\ ,\ )$

When $x = {}^+4$, $y^2 =$ ______. $({}^+4,\ )$ and $({}^+4,\ )$

When $x = {}^+5$ or $^-5$, $y^2 = 0$. $({}^+5,\ )$ and $({}^-5,\ )$

When $x = {}^+2$ or $x = {}^-2$, $x^2 = 4$ and $y^2 + 4 = 25$. So, $y^2 =$ ______. Therefore, $y = {}^+\sqrt{21}$ or $y = {}^-\sqrt{21}$. This gives us four more points: $({}^+2, {}^+\sqrt{21})$, $({}^+2, {}^-\sqrt{21})$, $({}^-2,\ )$, and $({}^-2,\ )$.

What number multiplied by itself gives 21 as the product?

$4 \times 4 = 16$ (too small)

$5 \times 5 = 25$ (too large)

It must be someplace between 4 and 5 or between $^-4$ and $^-5$. Let's try:

$4.4^2 =$ ______

$4.5^2 =$ ______

$4.6^2 =$ ______

Which is closest for y if $y^2 = 21$?

We are led to the conclusion that if $x = {}^{+}2$ then $y = {}^{+}\underline{\qquad}$ or ${}^{-}\underline{\qquad}$. This gives us the points $({}^{+}2, {}^{+}\underline{\qquad})$ and $({}^{+}2, {}^{-}\underline{\qquad})$.

And, if $x = {}^{-}2$ then $y = {}^{+}\underline{\qquad}$ or ${}^{-}\underline{\qquad}$. This gives us the points $({}^{-}2, {}^{+}\underline{\qquad})$ and $({}^{-}2, {}^{-}\underline{\qquad})$.

If, however, $y = {}^{+}2$ or $y = {}^{-}2$ then $y^2 = 4$

and $\underline{\qquad} + x^2 = 25$,

$x^2 = \underline{\qquad}$,

and $x = {}^{+}\underline{\qquad}$ or $x = {}^{-}\underline{\qquad}$.

From this we can locate four points:

$(\underline{\qquad}, {}^{+}2)$, $(\underline{\qquad}, {}^{+}2)$,

$(\underline{\qquad}, {}^{-}2)$, $(\underline{\qquad}, {}^{-}2)$

Are you quite sure the graph will be a circle no matter how many more points we locate?

Let's see what happens if $x = {}^{+}1$ or ${}^{-}1$.

If $x = {}^{+}1$ or ${}^{-}1$ then $x^2 = 1$,

$y^2 + \underline{\qquad} = \underline{\qquad}$,

$y^2 = \underline{\qquad}$,

$y = \underline{\qquad}$ or $\underline{\qquad}$.

What positive number multiplied by itself gives 24? It must be a little less than 5.

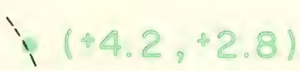
$4.8^2 = \underline{\qquad\qquad}$ (too small)

$4.9^2 = \underline{\qquad\qquad}$ (too large)

This provides enough information for four more points:

$(\underline{\ {}^{+}1\ }, \underline{\qquad})$, $(\underline{\qquad}, \underline{\qquad})$,

$(\underline{\qquad}, \underline{\qquad})$, $(\underline{\qquad}, \underline{\qquad})$

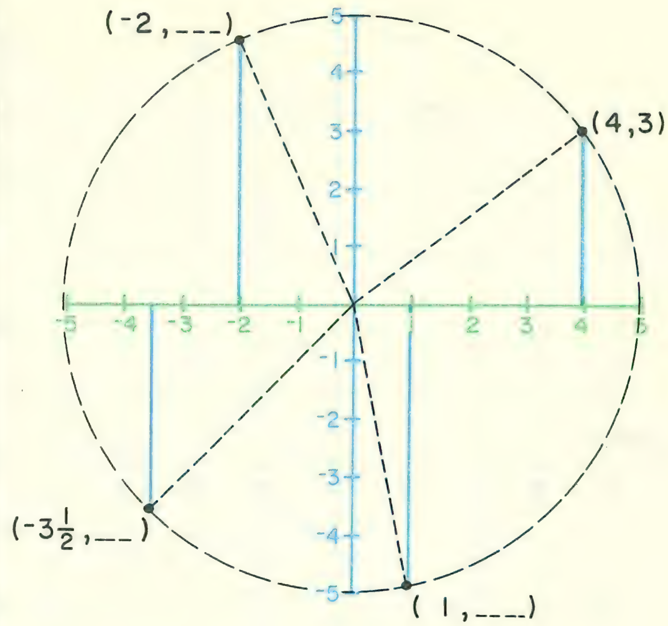


We have selected a point on the curve. It seems to have a location of $x = {}^{+}4.2$ and $y = {}^{+}2.8$. (These are only approximations.) We expect to find that

${}^{+}2.8^2 + {}^{+}4.2^2$ is about $\underline{\qquad}$,

and multiplying we have:

$\underline{\qquad} + \underline{\qquad}$ is $\underline{\qquad}$ (about $\underline{\qquad}$)



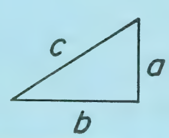
(To fill in the blanks, you need to compute only one bit of information. All the rest was done earlier.)

If from each point we locate, we draw a line perpendicular to the x-axis, a second line from that intersection to the center, and a third line from the point to the center, we have a triangle.

In each case, that triangle is a right triangle whose hypotenuse (or longest side) is a radius of our circle. Every radius is, of course, 5 units long.

Do you remember the Pythagorean Theorem?

In a right triangle, the square of the hypotenuse (c) equals the sum of the squares of the other two sides (a and b).



$$a^2 + b^2 = c^2$$

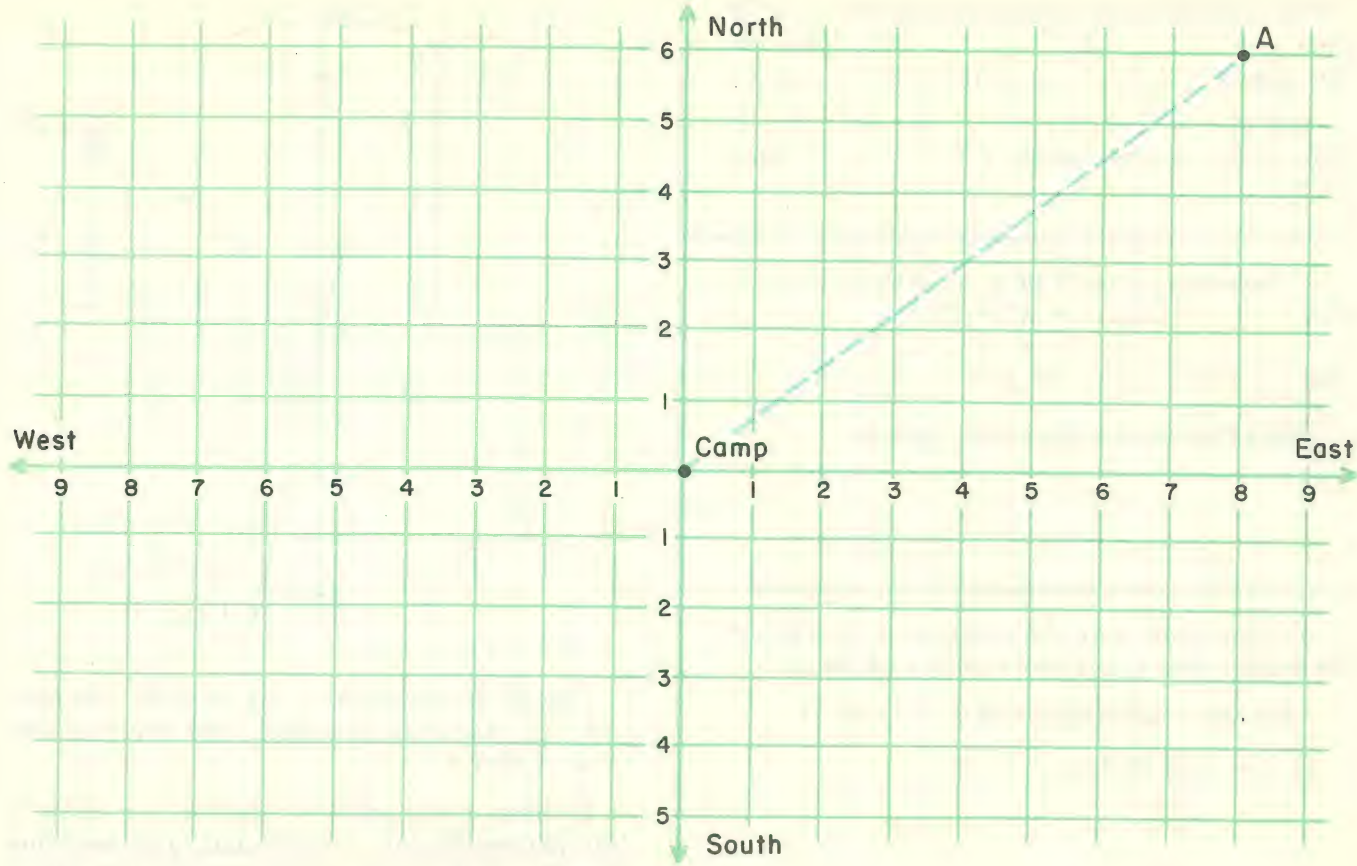
Compare: $y^2 + x^2 = 25$ and $a^2 + b^2 = c^2$

Can you imagine graphs for:

$y^2 + x^2 = 64$ $\quad$ $y^2 + x^2 = 7$ $\quad$ $y^2 + x^2 = 1$

or a graph for any equation of the form:

$y^2 + x^2 = n$?



A group of ten prospectors scattered from their camp. After a week's work, each reported his location with respect to the camp.

The radio operator at the camp made a map and marked the position of each man by placing a black dot and labeling it with the first letter of the prospector's name. The operator's notes indicated these positions:

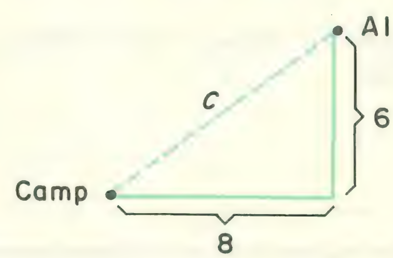
Al: 8 miles east and 6 miles north
Ben: 5 miles west and 2 miles north
Carl: 8 miles west and 6 miles north
Dan: 5 miles east and 5 miles north
Ed: 9 miles west and 5 miles south
Frank: 6 miles east and 5 miles south
Gil: 8 miles west and 2 miles south
Harry: 9 miles east and 2 miles north
Ira: 7 miles east and 3 miles south
Jack: 4 miles west and 5 miles south

Locate each prospector on the map and label the point as the operator did.

The director of the prospecting expedition asked the operator, "How far is Al from camp?"

"Al is 8 miles east and 6 miles north. I'll tell you in a minute the shortest distance by plane."

He drew the following sketch:



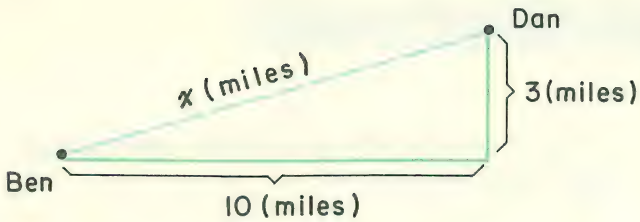
$c^2 = 8^2 + 6^2 = 64 + \_\_\_\_ = \_\_\_\_\_$

$c = \_\_\_\_\_$

"Al is _____ miles from camp," the radio operator advised.

"How far is Ben from Dan?"

"Well, Ben is 10 miles west of Dan and 3 miles south of him. I can tell you in a minute the distance between them — as a plane flies."



$x^2 = 3^2 + \_\_\_ = \_\_\_ + \_\_\_ = \_\_\_$

Now the operator has a problem. He must look for a number which, when multiplied by itself, or taken as a factor twice, gives 109.

$10 \times 10 = \_\_\_$ and $11 \times 11 = \_\_\_$

So he knows that the distance is more than ______ miles and less than ______ miles. He decides to try 10.5:

$$10.5 \times 10.5 = \_\_\_\_\_\_$$

The distance is ______ (*more* or *less*) than 10.5 miles. The operator makes the following calculations: ----------->

$10.3 \times 10.3 = \_\_\_\_\_\_$

$10.4 \times 10.4 = \_\_\_\_\_\_$

He knows that the distance between Ben and Dan is more than ______, but less than ______ miles. So, he makes the following calculations:

$10.43 \times 10.43 = \_\_\_\_\_\_\_\_\_$

$10.44 \times 10.44 = \_\_\_\_\_\_\_\_\_$

$10.45 \times 10.45 = \_\_\_\_\_\_\_\_\_$

$10.46 \times 10.46 = \_\_\_\_\_\_\_\_\_$

"I have found," the operator reported, "that Ben and Dan are more than ______ miles apart, but less than ______ miles apart. Do you want a closer approximation?"

"No," was the answer, "that is close enough — to the nearest hundredth of a mile."

"Please give me a record showing how far each man is from camp. Also, I wish to know how far each man is from the other — to the nearest hundredth of a mile."

All the distances are recorded as numbers of miles. 10.44^+ means more than 10.44 miles, but less than 10.45 miles apart.

Al	0										
Ben				10.44^+							
Carl											
Dan		10.44^+									
Ed											
Frank										10	
Gil											
Harry											
Ira											
Jack						10					
--to-->	Al	Ben	Carl	Dan	Ed	Frank	Gil	Harry	Ira	Jack	CAMP

Chapter THIRTEEN . . . THE AREA OF A CIRCLE

True or False?

(a) If two circles have the same radius then their areas are equal. T F

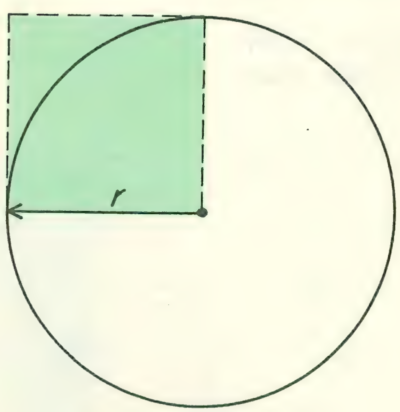
(b) If circle A has a larger radius than circle B then circle A has a larger area than circle B. T F

(c) If circle A has a smaller radius than circle B then circle A has a smaller area than circle B. T F

If you claim that all of these are true statements then you must believe that the area of a circle depends on its radius.

Since a radius is a segment, we cannot use it to measure area. Instead, we shall use the "square on the radius" as our unit of area.

We shall be looking for a ratio: the square on the radius is to the area of the circle as ___?___ is to ___?___.



$$\frac{r^2}{A} = \frac{?}{?}$$

How many tiles the size of r^2 ($r \times r$ or "r squared") would be needed to cover the region enclosed by the circle? ______

Could you place two whole tiles with size r^2 inside the circle so they would not overlap and no point on the tile would be outside the circle? ______

How many such tiles would you need to be sure to cover all points in the region enclosed by the circle? ______

So, we see that the area of the circle is more than r^2 and less than $4r^2$.

That's not a very close approximation.

Suppose that we use tiles that are $\frac{1}{2}r$ by $\frac{1}{2}r$—or tiles with an area of $\frac{1}{4}r^2$?

Notice Diagram I on the next page. This is one way this might be done. The tiles inside the green line are clearly "inside" the circle. The tiles inside the blue line clearly cover all points in the region enclosed by the circle.

Put a dot in every tile that touches any point on the circle. Complete the blue line so it includes all tiles that touch the circle at any point.

How many tiles are there inside the blue line in Diagram I? ______

Now we know that the area is somewhere between 4 of these smaller tiles and 24 of them.

Too small: $4 \cdot \frac{1}{4} r^2$, or r^2

Too large: $24 \cdot \frac{1}{4} r^2$, or $6r^2$

This is a very poor approximation, but it is only the beginning.

Now, suppose that we use still smaller tiles. Let's make each tile $\frac{1}{4}r$ by $\frac{1}{4}r$ — or $\frac{1}{16}r^2$.

Follow the same procedure as before in Diagram II.

How many tiles are inside the green line, tiles that do not touch any point on the circle?

______

How many tiles are used to cover all points in the region enclosed by the circle? ______

Now we know that the area is somewhere between ______ of these still smaller tiles and ______ of them.

Too small: ______ $\cdot \frac{1}{16} r^2 =$ ______

Too large: ______ $\cdot \frac{1}{16} r^2 =$ ______

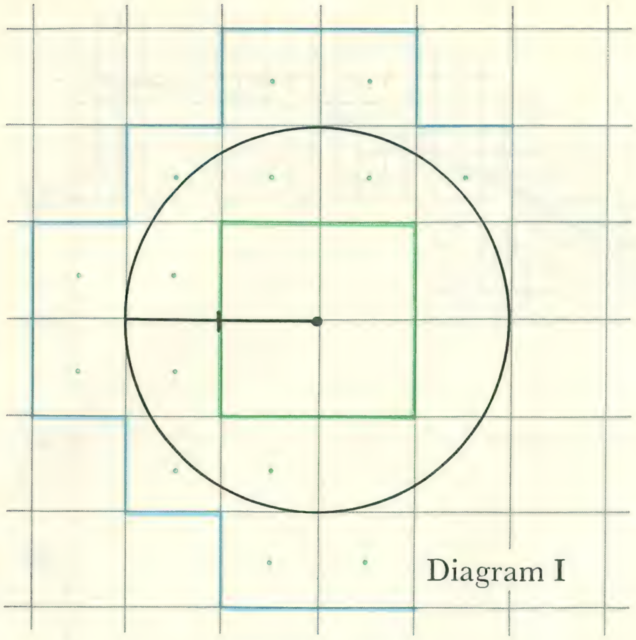


Diagram I

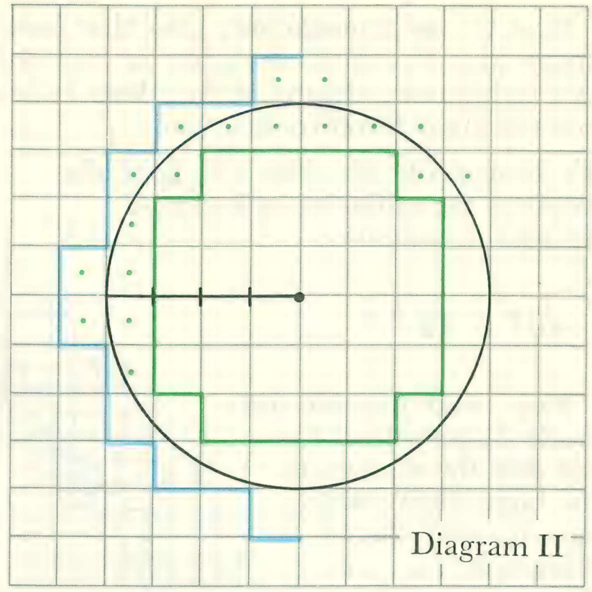


Diagram II

In Diagram III, we have tiles that are $\frac{1}{8}r$ on each side. So the tiles are

$$\frac{1}{8}r \cdot \frac{1}{8}r = \underline{\qquad\quad}.$$

How many tiles are there inside the green line? ______

Inside the blue line? ______

Too small: ______ $\cdot \frac{1}{64}r^2 =$ ______

Too large: ______ $\cdot \frac{1}{64}r^2 =$ ______

Now we need a magnifying glass. Diagram IV is seen as if it were twice size. (Only one-fourth of the diagram is shown. How can you know what the rest of the diagram would show?)

The radius is now divided into 16 parts, so each tile is $\frac{1}{16}r \cdot \frac{1}{16}r$, or ______.

How many tiles are there inside the green line? ______

How many are inside the blue line? ______

Too small: ______ $\cdot \frac{1}{256}r^2 =$ ______

Too large: ______ $\cdot \frac{1}{256}r^2 =$ ______

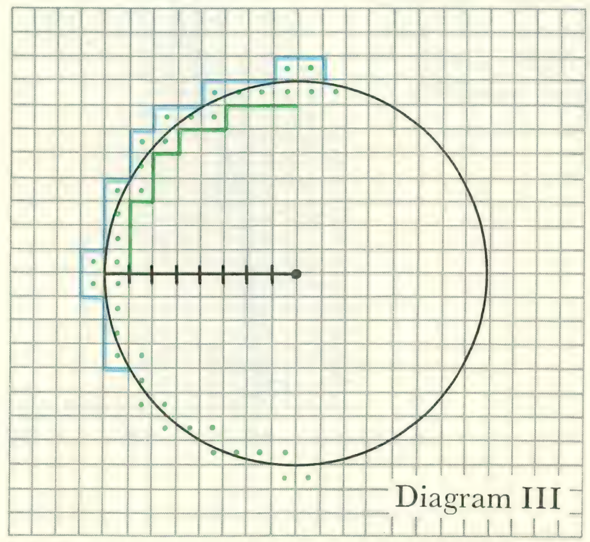


Diagram III

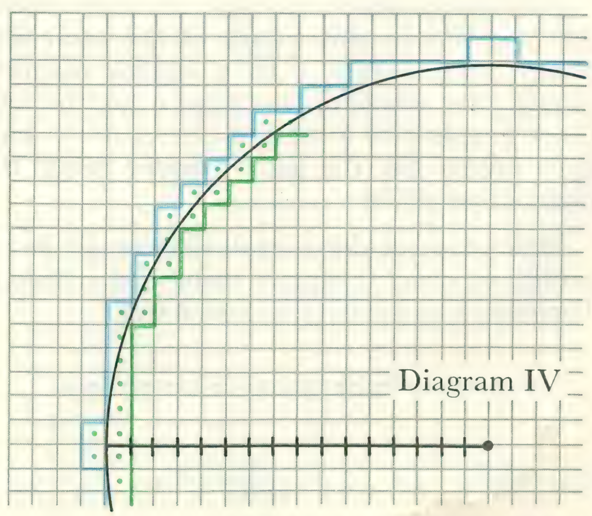


Diagram IV

(Under a magnifying glass)

Diagram	I	II	III	IV	V
Too small:					
Too large:					

(See the next page for Diagram V.)

Next we use a magnifying glass that makes things look five times as large as they are. Our radius now seems to be five times as long as it was in the first three diagrams.

We have made the tiles only $\frac{1}{40}$ of the length of the radius on each side, so the area of each tile is

$$\frac{1}{40}r \cdot \frac{1}{40}r = \_\_\_\_\_\_$$

How many tiles are there inside the green line? (You can find the number in the large blocks without drawing them.) There are ______ in this diagram of $\frac{1}{4}$ of a circle. In a diagram of the whole circle, there would be ______

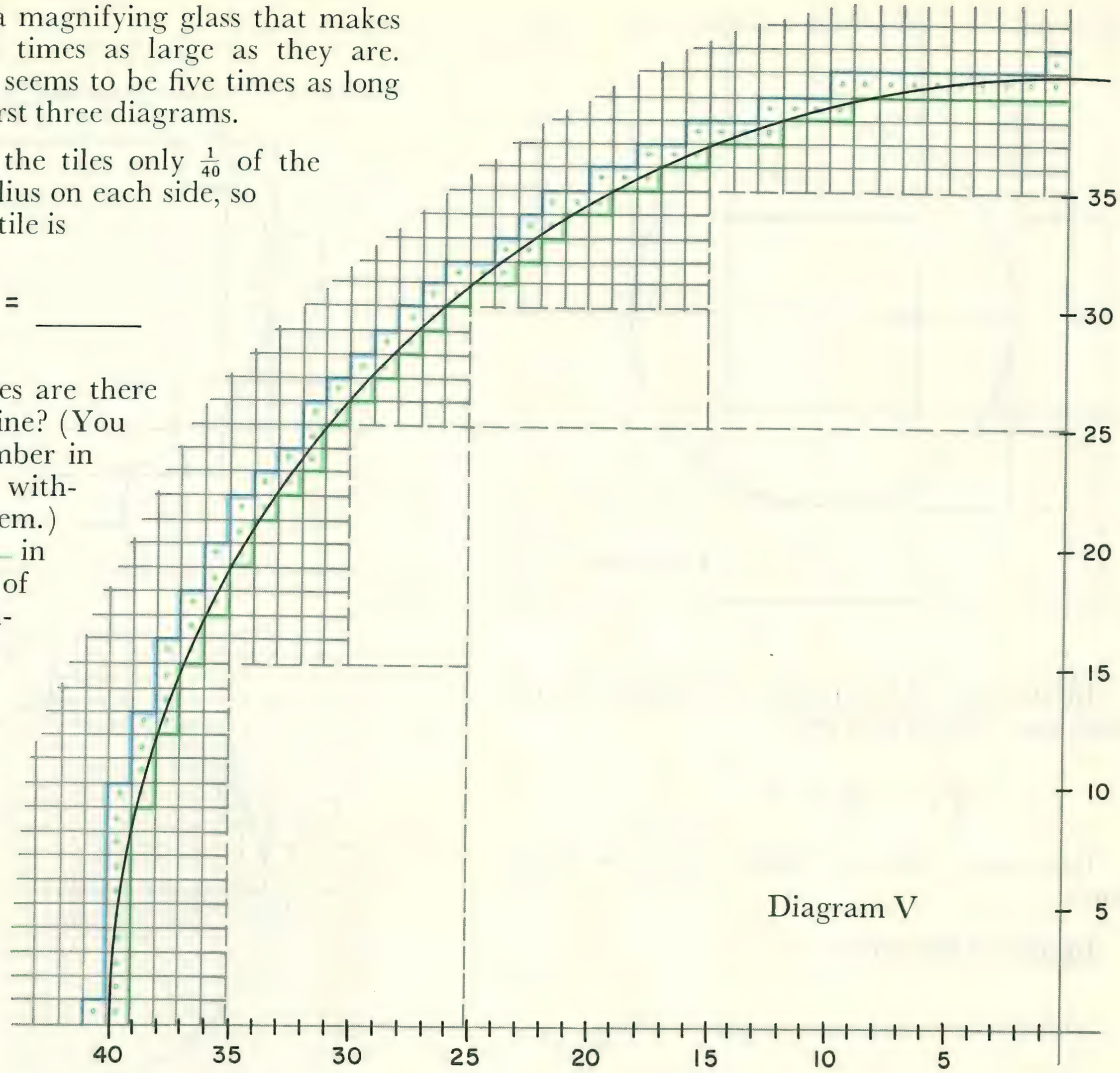


Diagram V

How many more than that are there inside the blue line in this fourth of the total diagram? ______ more. In a similar diagram showing the whole circle, there would be four times as many, or a total of ______ more blue than green. So, the total number inside the blue line would be ______.

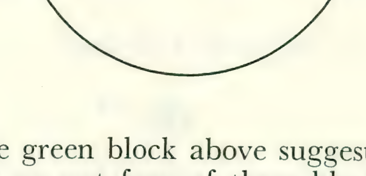
How many of these small tiles would be required to cover a square using the radius in Diagram V as the length of each side?

____

Too small: ______ $\cdot \frac{1}{1600} r^2 =$ ______ r^2

Too large: ______ $\cdot \frac{1}{1600} r^2 =$ ______ r^2

(Record these results in the chart on the previous page.)

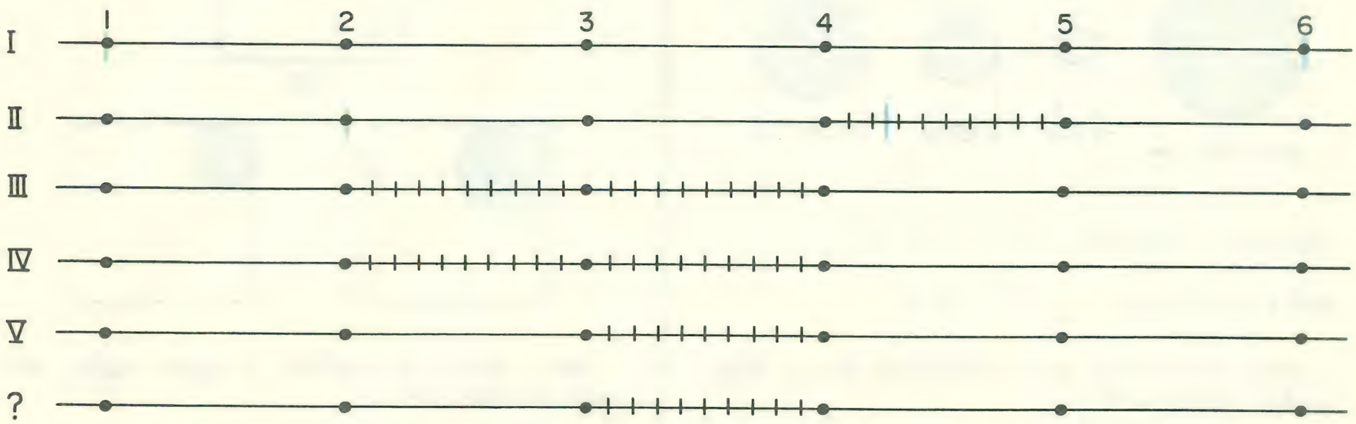


The green block above suggests our idea of r^2. If we cut four of these blocks into 1,600 tiny squares each, we know we could put at least ______ of the tiny squares completely inside our circle. Also, we would not need more than ______ of the tiny squares to cover the region enclosed by the circle. Since each tiny square has an area of $r^2 \div 1600$, the area of the circle is somewhere between ______ r^2 and ______ r^2.

Express the results of the five studies in decimals and record below . . . (to the nearest hundredth)

	I	II	III	IV	V	?
Too small	$1.0r^2$					
Too large	$6.0r^2$					

What do you notice? Now, indicate these results on a series of number lines.



We could keep magnifying our diagrams and divide the radius into the smallest units we could handle.

In each experiment, the difference between the area that is clearly "too small" and the area that is "too large" would become less and less.

Could we design an experiment with such a large radius and such small divisions that we would reduce that difference to zero? ______

As we kept recording our continuously magnified experiments on a number line, would they crowd in from both directions on a certain point? ______

We know that such a point must be located somewhere between the points labeled ______ and ______. What point is half-way between? ______

Our experiments have led us to believe that the point 3.14 is very close to the number we're looking for.

It should come as no surprise that more careful experiments will lead us closer and closer to a ratio we have encountered before: π — an irrational number, for which the approximations $3\frac{1}{7}$ and 3.1416 are commonly used.

We can note this relationship between the radius and the area of a circle as:

Area = πr^2

and the relationship between the circumference and radius as:

Circumference = $2 \pi r$

If we were to express each in terms of the diameter, d, we could write:

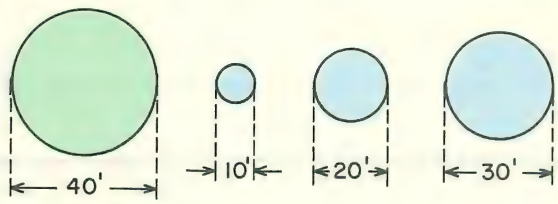
Area =

Circumference =

John Hayes, a gardener, said that he had in one day planted a round flower bed that was 40 feet across.

"That's nothing," Bill Jones said. "Today I planted three flower beds. One was 10 feet across; another was 20 feet across; and the third was 30 feet across."

John Hayes scratched his head. "Well, that's still not as much" . . . and an argument followed. Can you settle it?

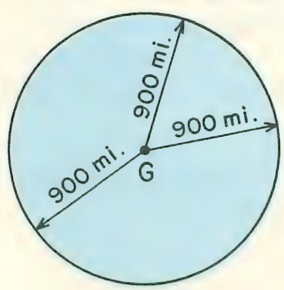


John Hayes planted ________ sq. ft.

Bill Jones planted ________ sq. ft.

(Use any of the approximations for π that we have discussed.)

When America's first astronaut, Col. John Glenn, was in orbit in 1962, the newspapers reported that he and his capsule were often about 100 miles above the earth, and that he could see about 900 miles in all directions. Of course, mountains would sometimes cut off his vision and the curvature of the earth was involved; but we can make a rough estimate of the size of the region he could see.



Col. Glenn could see about

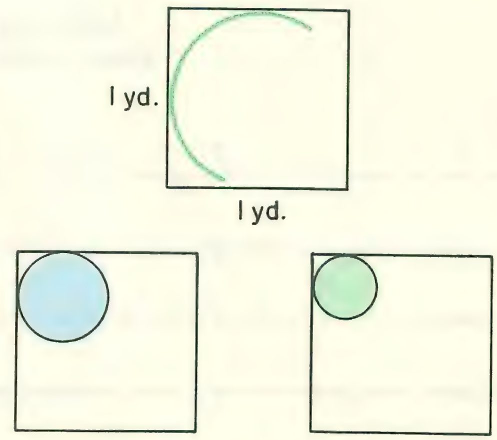
____

Mary said that if pizza pies were the same thickness, you would get the same amount in a pie that measured 14″ across as in four pies that measured 7″ across. Was she right?

____

Sarah had three pieces of cloth, each 1 yard square. From one, she wished to cut the largest possible circle. From the second, she wished to cut four circles the same size and as large as possible. From the third, she wished to cut nine circles the same size and as large as possible.

Finish the sketches below that suggest the situation.



How much of Sarah's 3 square yards was used for the circles?

____

In which case was there the most scrap cloth?

____

In the first case, when Sarah cut out only one large circle, how much of the 1 square yard piece was scrap?

____

In which case was there the least scrap?

____

How much scrap was there altogether?

____

If Sarah had another piece of cloth 1 yard square and needed two circles the same size and as large as possible, what would the area of each circle be?

____

How much would be scrap? ________

Explore the problem Sarah would face if she wished to cut three circles as large as possible and all the same size from a 1 yard square of cloth.

Mr. Bowen bought 100 feet of fence to protect his garden. He wanted a garden that was rectangular in shape and as large as he could enclose with 100 feet of fencing.

He made his garden the shape of the green outline shown here — 30 feet long and 20 feet wide. Did he use all his fencing?

After he was finished, his neighbor said, "You could have made your garden larger if you had used other dimensions."

"But I had only 100 feet of fencing," Mr. Bowen answered, "and I wanted all the corners square."

"I still think I could enclose a larger area," the neighbor replied. Could he? Mr. Bowen made the following study:

length in feet of each of 2 opposite sides	30	35	40	45	25	23	21	27	28
length in feet of each of other 2 sides	20	15	10	5	25				
square feet of area enclosed	600								

He selected the three best examples and extended his study:

length in feet of each of one pair of sides	23	24	24.5	24.8	25	25.1	25.3	25.7	27
length in feet of each of the other pair	27				25				23
number of square feet of enclosed area									

This result interested Mr. Bowen, and he explored the idea. Before beginning, he worked out some shorthand to help him simplify his work. ------------------------→

A is the number of square feet of area.
l is the number of feet in length.
w is the number of feet in width.

$l \cdot w = A$

"Suppose that I had 200 feet of fencing."

l	25	35	45	50	55	53	51	51.5
w	75							
A								

"Suppose that I had only 90 feet of fencing."

								?
l	40	30	25	20	24	22	23	
w								
A								

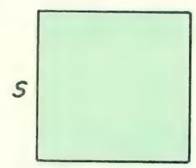
Mr. Bowen still was not thoroughly convinced. He tried similar studies, considering the problem each time with a different amount of fencing.

What do you think he decided when he considered having 121 feet? The largest enclosure would be ______ feet long and ______ feet wide. The area enclosed would thus be ______ square feet.

Chapter FOURTEEN . . . PROBLEMS ABOUT AREA AND VOLUME

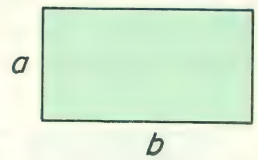
Letters in each sketch are known dimensions. Write a formula for finding the area of the green region in each sketch.

(1) a square



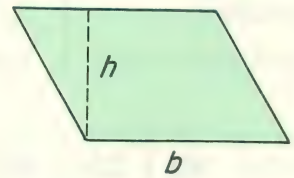
Area =

(2) a rectangle



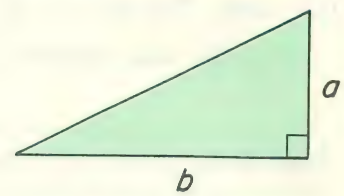
A =

(3) a parallelogram



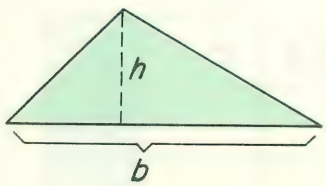
A =

(4) a right triangle



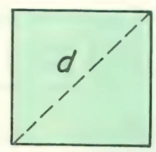
A =

(5) any triangle



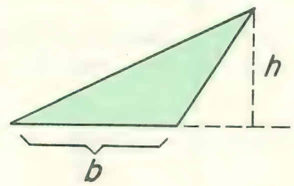
A =

(6) a square



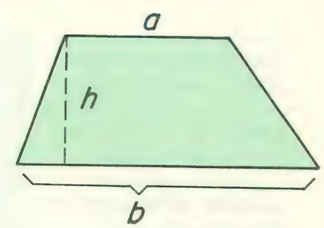
A =

(7) any triangle



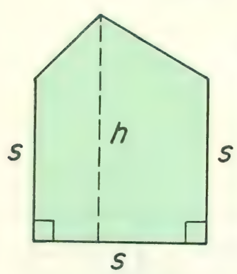
A =

(8) a trapezoid



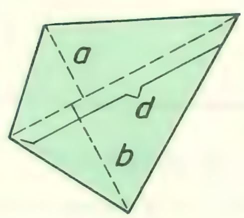
A =

(9) a square with a triangle on top



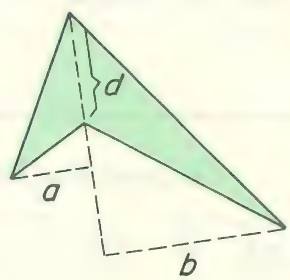
A =

(10) any quadrilateral (convex)



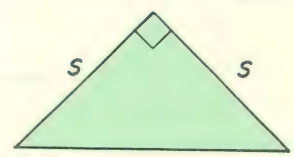
A =

(11) any quadrilateral (concave)



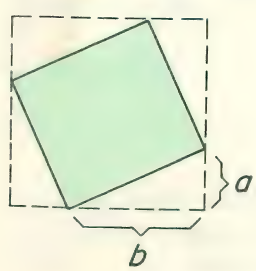
A =

(12) a right isosceles triangle



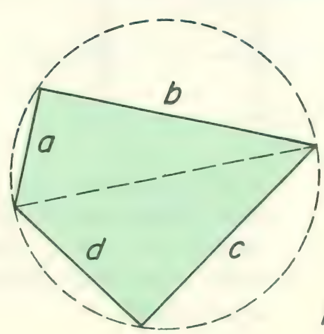
A =

(13) a square within a square



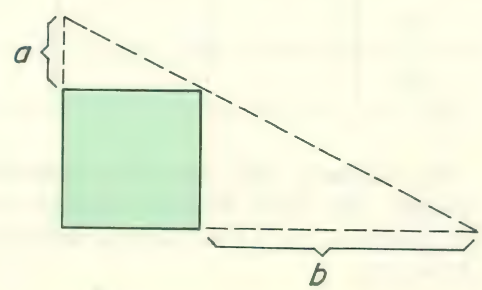
A =

(14) a quadrilateral in a circle — with two opposite vertices 180° apart



A =

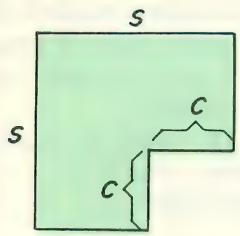
(15) a square in a right triangle



A =

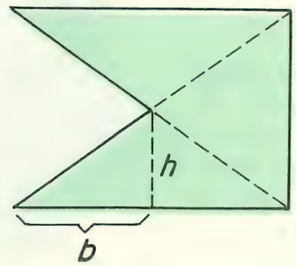
Letters in each sketch are known dimensions. Write a formula for finding the area of the green region in each sketch.

(16) a square with a square corner removed



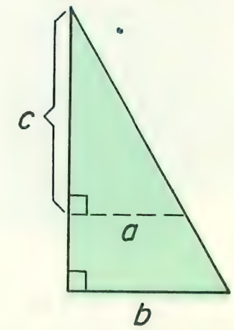
A =

(17) a rectangle with a wedge cut out



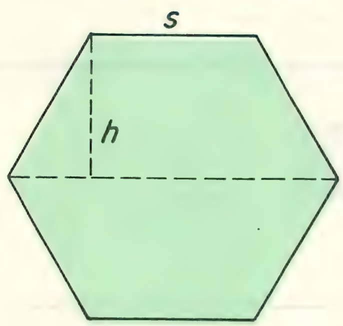
A =

(18) a small right triangle covering part of a large one



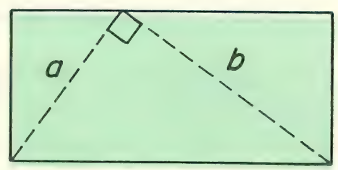
A =

(19) a regular hexagon



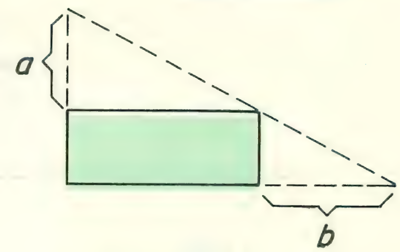
A =

(20) a right triangle in a rectangle



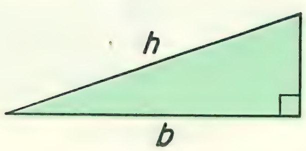
A =

(21) a rectangle in a right triangle



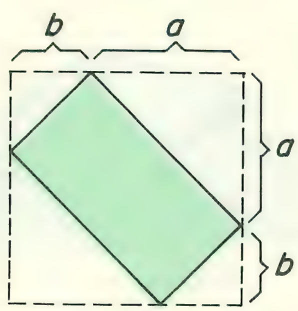
A =

(22) a right triangle



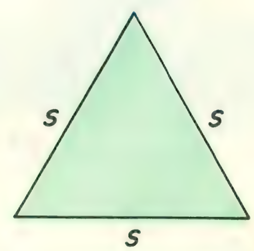
A =

(23) a rectangle in a square



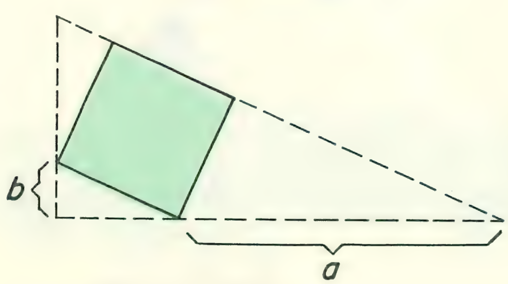
A =

(24) an equilateral triangle



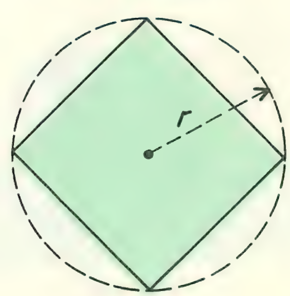
A =

(25) a square in a right triangle



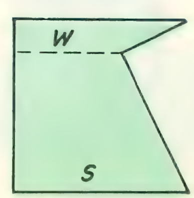
A =

(26) a square in a circle with radius r



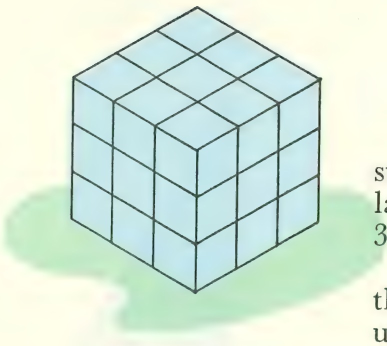
A =

(27) a square with a right triangle removed



A =

a cube

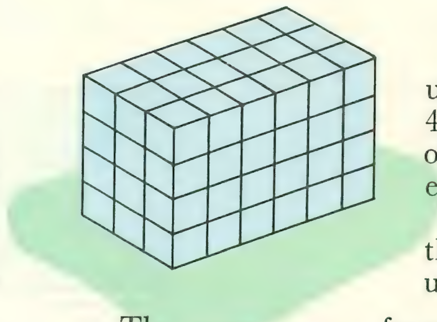


Here are unit cubes stacked in a pile: 3 layers, with 3 rows of 3 cubes in each row.

In such a stack there would be ____ unit cubes.

Using a face of a unit cube as a unit of area, each face of the stack above has ______ units of area. There are 4 sides, a top, and a bottom — or a total of ______ faces, and the total area of those faces is ______ units of area.

a rectangular solid



Here is a stack of unit cubes stacked in 4 layers with 3 rows of ______ cubes in each row.

In such a stack there would be ____ unit cubes.

There are ______ faces to the rectangular solid, and their combined area, or surface, is ______ units of area.

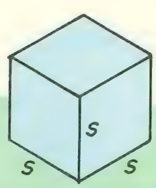
A problem for your imagination: How many unit cubes in the stack above could not be seen at all from any angle, no matter how you turned the stack? ______

Below, we shall think of the edge of a unit cube as a unit of length, the face of a unit cube as a unit of area, and a unit cube itself as a unit of volume.

Letters are known dimensions.

---

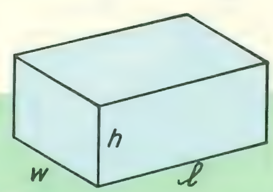
a cube



Volume = S^3

Surface =

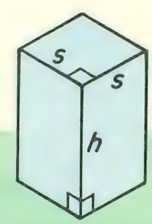
a rectangular solid



Volume =

Surface =

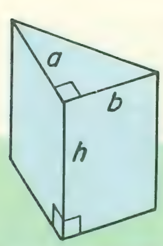
a right prism with a square base



Volume =

Surface =

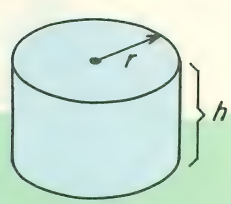
a right prism with a right triangle as base



Volume =

Surface =

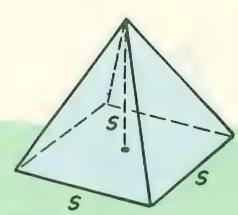
a right circular cylinder



Volume =

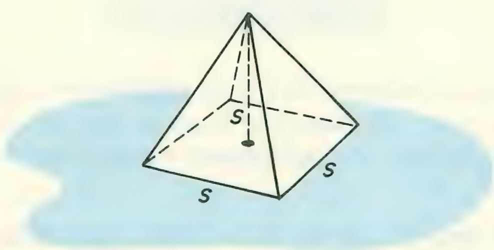
Surface =

a regular pyramid with a square base



Base is *s* by *s*.
Height is also *s*.
Make a guess about its volume.

Volume of a Regular Pyramid with a Square Base.

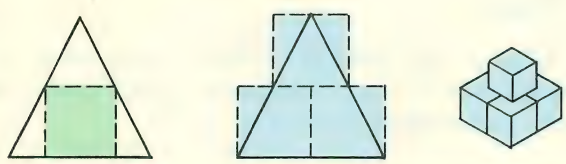


Imagine that this pyramid is made out of a cube that is s by s by s. The peak, or vertex, would be a point on the top of the cube.

What is its volume?

Suppose that s *is 1.* All we know is that the volume is more than 0 and less than 1.

Suppose that s *is 2.* We could fit 1 full unit cube inside and we could have made the pyramid out of 1 unit cube placed on top of 4 unit cubes — or a total of 5 unit cubes. Here is a sketch to suggest these ideas:

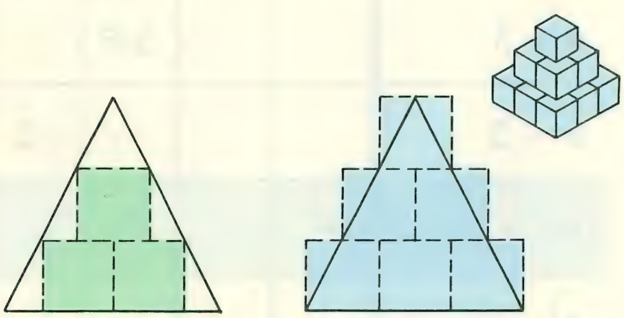


1 fits inside

pyramid fits inside 5 cubes

We know that if $s = 2$, the volume is more than ______ and less than ______.

Suppose that s *is 3.*

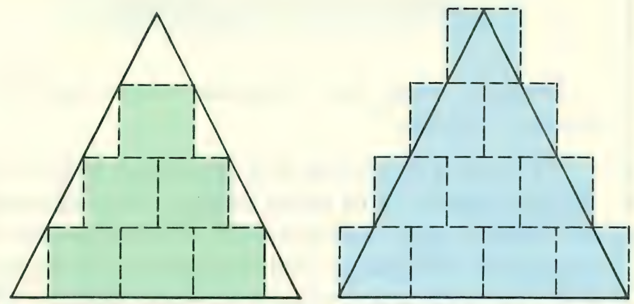


a stack of 4 + 1 fits inside

pyramid fits inside a stack of 9 + 4 + 1

We know that if $s = 3$, the volume is more than ______ and less than ______.

Suppose that s *is 4.*



a stack of ____ + ____ + ____ fits inside

pyramid fits inside a stack of ____ + ____ + ____ + ____

We know that if $s = 4$, the volume is more than ______ and less than ______.

A pattern has emerged; so you could extend the experiments without any sketches.

Let's summarize our results thus far and extend the pattern a little.

if s is	volume is $>$	volume is $<$	average of $>$ and $<$	s^3	$\frac{\text{avg.}}{s^3}$
1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$
2	1	5			

The fractions in the column on the right are coming closer and closer to a very simple fraction. The last entry in that column is just $\frac{1}{216}$ more than ______!

So, we can feel quite sure in guessing that the pyramid has ______ the volume of the cube it is made from. The same relationship holds for any pyramid which is cut in this way from a rectangular solid. It also holds for any circular cone which is cut in this way from a right circular cylinder.

Arithmetic By Space Phone

Imagine that the telephone rings and you answer, "Hello."

"This is X-Nighyun in a spaceship near your Earth's moon. I'm from Mars. Alec Marson, my friend, once visited earth. He taught me to speak your language and told me to phone you if I ever got near enough to your planet. He said you would tell me about arithmetic. Do you know anything about arithmetic?"

What would be your answer?

"Well . . . well . . . Say, can you call me back tomorrow?"

"What's counting?"

"Well . . . well . . . Say, can you call me back tomorrow?"

"I certainly shall."

Puzzled, you say goodbye to X-Nighyun and hang up the phone. Questions buzz in your head. There's so much to explain, and so little to build on. How will you ever explain fractions, or even multiplication?

Plan of Attack

You've learned so much arithmetic over the years that you wonder how it can all be explained over the telephone even if you had a year.

So you decide on an unusual plan of attack. There is a way to teach X-Nighyun about counting, addition, and multiplication all in one easy lesson. When you are done, he will be able to count, add, and multiply — though he may not understand at all what he is doing. You might name this lesson:

Arithmetic Without Understanding

So, the next day you tell X-Nighyun to write down a whole string of symbols that are all different from each other.

"What kind of symbols?"

"I don't care — just make them all different."

"Done."

"Begun, you mean! When we run out of symbols you've written down, you'll have to invent some more.

What Are My Rules?

A.	I	II	III	IV
a	5	-5		
$a+7$	12		$7\frac{1}{2}$	
$a-4$	1			
$2a$	10			14.6
$2a+a$	15			
$2(a+a)$	20			
a^2	25			

B.	I	II	III	IV
x	7	$2\frac{1}{2}$		
$x+3$			3.8	
$x-3$				-3
$(x+3)(x-3)$	40			
x^2				
x^2-3^2	40			
$15-3x$	-6			15

"Now, call the first symbol you wrote 'zero,' the next one 'one,' the next one 'two,' etc.

"Next, make two charts with many rows and columns. Above each column, write the symbols in order — zero, one, two, three, etc. — one to a column.

"At the left of the first column, write the symbols in order, beginning with zero, one to each row.

"Invent a symbol or mark for addition and another for multiplication. Put one of the marks on each table."

"But what is this addition and multiplication all about?"

"You don't need to know."

While you don't know what symbols X-Nighyun has invented, he has charts equivalent to these:

+	0	1	2	3
0	0			
1				
2				
3				

X	0	1	2	3
0	0			
1				
2				
3				

"In the block in row 0 and column 0, write a 0.

"In the chart marked 'addition,' start filling in the blocks this way: As you move along a row or down a column away from the labeled end, write the next symbol from your list."

"Done. What about the other chart?"

"Select any block. It will be in a row and in a column. Now count the blocks which are *both* above the row *and* between that column and the left-hand edge."

A.	I	II	III	IV
r	2			
$3r+7$	13	37		
$3(r+7)$	27		21	
$20-4r$	12			10
$4(5-r)$	12			
r^2-r	2			
$r(r-1)$	2			
$3r^2$	12			
$(3r)^2$	36			

B.	I	II	III	IV
b		-5		
$b-1$			.2	
$(b-1)^2$				
b^2-1^2	63			
$(b+1)(b-1)$				-1
$4b-b$	24			
b^2-2b	48			
$b(b-2)$				
$b(2-b)$				

"Count?"

"Yes. For each such block, start with your symbol 'one' and write the list in order, one in each block. When you've filled them all, copy the last one in the block you first selected."

"What if there are no blocks *both* above the row *and* between the column and the left-hand edge?"

"Then write a 0 in that block."

X-Nighyun must have made a small chart — or he works with all hands at once — because he soon called out,

"Done!"

"Let's check your work. In the first row of your addition chart and in the first column, you should have written the string of symbols you invented and in the same order."

"Exactly."

"You can check each row in the multiplication chart in this way. The top or first row consists of zeroes. The next row is the string of symbols you invented and in the original order.

"For the next row, start with the 0 in your string and mark every other one starting 0, 2, 4, etc.

"For the next row, do the same but make each jump one symbol more than the jumps for the preceding row.

"Follow the same procedure for each succeeding row."

"Done."

"So you are ready to add and to multiply. "Three plus four" is an instruction that says: Find your symbol 3 at the top of a column in the addition chart and find the 4 at the end of a row. Find the symbol you wrote in the block in that row and column. It is the 7. So, we report that

$$3 + 4 = 7.$$

A.	I	II	III	IV
a	$2\frac{1}{2}$		-8	
$2a$	5			
$2a + 7$	12	8		
$4a + 14$	24			14

B.	I	II	III	IV
c				
$2c$	19			
$2c - 7$		-7	-17	
$c - 3\frac{1}{2}$	6			$33\frac{1}{2}$

C.	I	II	III	IV
d	3			
$d + 4$	7	$5\frac{2}{3}$		
$3(d + 4)$	21		100	
$3d + 12$	21			9

D.	I	II	III	IV
x	7			
$8 - x$	1		8	
$2(8 - x)$	2	-184		
$16 - 2x$				15.9

"If I had said 3 times 4 or 3 multiplied by 4, you would use the multiplication chart and report your findings as:

$$3 \times 4 = 12$$

and that's all there is to addition and multiplication."

"You mean it's nothing more than looking something up in one of two charts?"

"Exactly!"

"But my charts aren't big enough to include all my symbols along the top and down the side."

"That's your problem. You know how to make your charts bigger if you need to."

"I guess you're right. But what's the point of playing such a game? Isn't there any use to arithmetic?"

"One thing at a time, please, one thing at a time. Now at least you know how to play the games of adding and multiplying.

"At least you know that no matter what pair of symbols you decide to add or multiply, there is a symbol that we call their sum in addition or their product in multiplication.

"If a and b are any two symbols you choose from the list, we know there must be symbols c and d which are their sum and product.

"We express that idea this way: For every a and every b, there are a c and a d such that

$$\boxed{a + b = c} \text{ and } \boxed{a \times b = d}.$$

"Where we can say that, we say that we have defined addition and multiplication."

So much for arithmetic without understanding!

A.	I	II	III	IV
m	10	$\frac{1}{3}$		
$\frac{m}{2}$	5		.05	
$\frac{m}{2} + 8$	13			
$3\left(\frac{m}{2} + 8\right)$	39			9

B.	I	II	III	IV
r				
$2r$	12	1		
$\frac{2r}{3}$	4		.8	
$\frac{2r}{3} + \frac{1}{2}$				$^{-}\frac{1}{6}$

C.	I	II	III	IV
n	17			
$n - 3$		20		
$\frac{n-3}{2}$	7	10		
$\frac{n}{2} - \frac{3}{2}$	7		-4	0

D.	I	II	III	IV
s	4			
$s + 5$		17		
$\frac{s+5}{3}$			1	
$\frac{s}{3} + \frac{5}{3}$				$2\frac{1}{2}$

Patterns in the Charts

X-Nighyun reports the next day:

"I played your game with the charts. The charts interest me most. For every row in either one, there is a column exactly like it. Because of this, I could flip my chart around the diagonal from the "0,0" corner and the chart is unchanged (except my symbols get twisted on their sides). And this answers a question I had: When you select a pair of symbols, which one do you find above the columns and which at the end of the rows? It turns out that you get the same result one way as the other."

"Right!" you break in. "So we say that, for any pair of selections a and b,

$$a + b = b + a \quad \text{and} \quad a \times b = b \times a.$$

"We refer to these ideas as the commutative principle for addition and the commutative principle for multiplication."

"Big name for such a simple idea! But let me continue.

"There's a row and a column in each chart that is the same as a row and column in the other chart. Further, it is the row that lists all my symbols in order — zero, one, two, etc. For the addition chart, the zero row and the zero column; for the multiplication chart, they are the one row and the one column."

"Right again," you interrupt. "We say that 0 for addition and 1 for multiplication are 'identity elements' because, for every a,

$$a + 0 = a \quad \text{and} \quad a \times 1 = a.$$

"Identity elements, eh! More fancy language for a simple idea.

"I also notice that in the multiplication chart, the entire zero row and zero column are the same — all zeros — and zero appears nowhere else."

A.	I	II	III	IV
d				
$2d$				
$d + 5$	12			
$2(d + 5)$		4		
$2d + 10$	24			
d^2	49		$\frac{1}{4}$	
$3d + 2$	23			3.2
$10 - d$	3			
$-1d$	-7			

B.	I	II	III	IV
k				
$k + 6$				
$k - 6$				-5.5
$(k+6)(k-6)$				
k^2				
$k^2 - 6^2$			-36	
$3k - (k+2)$		10		
$2k - 2$				
$2(k - 1)$	18			

"We've noticed that, too. We say that, for any a and b,

$$a \times b = 0 \text{ if and only if } a = 0 \text{ or } b = 0.$$

"Nice way to put it! . . . and that's all the homework I did.

"Wait! I have a question. Can you only play your games with a pair of symbols? Seems to me that would get boring much too soon."

"Good question!

"Write down some of your symbols on separate pieces of paper. Label one a, another b, and another c.

"Find the sum of a and b; then find the sum of your result and your c selection."

"Done."

"Start over. Find the sum of b and c; then find the sum of a and your sum of b and c.

"Done! Say, is it an accident that I got the same final result in both cases?"

"Do you want me to tell you?"

"No. Give me a moment or two."

A moment or two later, X-Nighyun was back on the phone.

"It works every time! I tried using the same selections on the multiplying chart, and got the same results no matter which pair I selected first."

"We refer to this idea in this way: For every a, b, and c,

$$(a + b) + c = a + (b + c) \quad \text{and}$$

$$(a \times b) \times c = a \times (b \times c).$$

"We have fancy names for this idea, too: the associative principle for addition and the associative principle for multiplication."

"I'm amazed. All of this works out and you don't even know what my symbols look like."

A.	I	II	III	IV
a				
$2a$	22			
$2a + 8$		$9\frac{1}{3}$		
$\frac{2a+8}{3}$			0	$-4\frac{2}{3}$

B.	I	II	III	IV
t				
$t + 7$				
$2(t+7)$	25		15.4	
$\frac{2(t+7)}{5}$		6		2

C.	I	II	III	IV
x				
$2x$				
$-2x$				
$15 - 2x$	7	-3	21	14

D.	I	II	III	IV
y				
$-3y$				
$-3y + 12$				
$\frac{-3y+12}{2}$	-9	9	6	$4\frac{3}{4}$

"So far, we have discussed ideas that apply as well to one chart as to another — except for the row and column of zeros in the multiplication table.

"Now I'm going to show you that the charts can be used together. Again, select some symbols and label them *a*, *b*, and *c*.

"First, add *a* and *b*. Multiply this sum by *c*. Note the product.

"Second, multiply *a* by *c*, then multiply *b* by *c*, and add the products.

"Surprised?"

"It can't be an accident that I got the same result in both examples! Let me change my selections . . . it works again. My guess is that you've got a fancy name for that one."

"You're right. We refer to the idea this way: for every *a*, *b*, and *c*,

$$(a+b)\times c=(a\times c)+(b\times c)$$

and we call it the distributive principle for multiplication over addition."

"The game is getting more interesting. That was a real surprise."

At this point you explain that the tables can be used in reverse as a "subtraction table" and a "division table."

"Pick a pair of symbols, and label with an *a* the one that comes further along in your string. Label the other with a *b*. Now, look for *a* in the blocks of the addition chart. You'll find a whole diagonal full of *a*'s. Pick the *a*-block that's in the column headed by a *b*. This block is in the row which you can label with a *c*. We record this as:

$$a-b=c$$

A.	I	II	III	IV
a	6	1		.3
$2a$	12	2		.6
	13	8	$7\frac{1}{2}$	
		16		14.6
a^2	36		$\frac{1}{4}$	
		3		.27
	30	0	$\frac{-1}{4}$	
	19	24		24.7
	$9\frac{1}{2}$	12	$12\frac{1}{4}$	

B.	I	II	III	IV
y	5		1	1.2
	25	81	1	
	30	90	2	
y – ----	-5	-1	-9	
	125		5	
	$12\frac{1}{2}$	$40\frac{1}{2}$	$\frac{1}{2}$	
	$62\frac{1}{2}$	$202\frac{1}{2}$	$2\frac{1}{2}$	3.6
	36	100	4	4.84
	50	158	6	6.68

We see that the *a* selection cannot come in X-Nighyun's string of symbols before his other (*b*) selection. And we agree with him that this requirement puts a serious limit on "subtraction."

We explain division in a similar fashion and again agree with him that there is a serious limit on "division."

$$a \div b = c$$

"And what's more," we point out, "we have to rule out dividing by 0. We shall discuss this later."

"So, whenever you make selections and add or multiply, you can then start with the sum or product and use either of the initial selections and subtract or divide. I'll put it this way: For every *a*, *b*, and *c*,

if $a + b = c$ then $c - b = a$ and $c - a = b$;

if $a \times b = c$ then $c \div b = a$ and $c \div a = b$."

"Very good . . . unless *a* or *b* is zero. In that case, you must modify the second part of your statement."

From Symbols to Numbers

"Let's talk about numbers."

"Not symbols?"

"No, numbers!"

"What are numbers?"

You knew this would happen; so you're prepared.

"Do you have hands?"

"Yes, I am using a hand to guide my space ship, another hand to hold the sandwich I'm eating, another hand to hold the book I'm reading, and my other hand is in my pocket. . . . Sure. I have four hands."

"That's four hands and four is a number."

"Do you mean that numbers are hands and hands are numbers?"

"No! No! No! No! . . . You see, I said 'no' four times." You tap the telephone with a pencil — tap, tap, tap, tap. "That's four noises. Boom! Boom! Boom! Boom! That's four booms . . . a boom for each of your hands

A.

	I	II	III	IV
m	6	-1		9
	18			27
	36	1		81
	23	16	17	

B.

	I	II	III	IV
x	7	.7	$\frac{1}{3}$	4
	93	99.3		96
	36			42
	42		$-\frac{2}{9}$	12

C.

	I	II	III	IV
n		17	$\frac{1}{2}$	5
	199	33	0	9
	301	52	$2\frac{1}{2}$	16
	20,000			50

D.

	I	II	III	IV
b	5	8	-2	
	24	63		$1\frac{1}{4}$
	50		-20	15
	1	$1\frac{3}{5}$		$\frac{3}{10}$

. . . No, No, No, No . . . tap, tap, tap, tap, Boom, Boom, Boom, Boom . . . hand, hand, hand, hand . . . four hands, four booms, four taps, four no's. . . ."

"Aha! . . . and you gave me four examples. If there's enough for each hand and no more, that's four. Now I can tell you that I have four fingers on each hand."

"Another number is one . . . you have one hand in your pocket; Earth has one moon; our planet revolves around a single sun; you are one Martian."

"Herchoo," X-Nighyun explodes over the phone, "and I suppose that is one sneeze."

"Yes!"

"One 'yes.' "

This could go on for a long time, so you get on with your job.

"Tie a string around one finger of the hand you have in your pocket."

"One string around one finger."

"Now tie another string around another finger on that hand."

"Completed!"

"Two is another number . . . two pieces of string, two fingers with string on them . . . Boom, Boom. . . ."

"Say, then on that same hand I now have two fingers that do not have strings on them. Right?"

"Right. Three is another number. You have three hands that don't have any strings on any of the fingers. . . . Hand, hand, hand."

A.	I	II	III	IV
a	5	$\frac{1}{2}$		-3
b	8		8	
$a+b$		$\frac{3}{4}$	9.5	7
$a+2b$	21			17

B.	I	II	III	IV
w				
x	9			
$w+x+5$		6.4	5	5
$w-x$	14	0	8	-8

C.	I	II	III	IV
c				
d	.7			
$c+d$	3.7	100	19	1
$c-d$		52	-5	0

D.	I	II	III	IV
y	3		$\frac{1}{5}$	
z				
$y \times z$	21	72	$1\frac{1}{5}$	5.1
$y+z$	10	17		17.3

"Three, four, one, two — that's four numbers. Is that all the numbers there are . . . as many numbers as I have fingers on one hand?"

"No. There is a number for every symbol you invented, no matter how many you care to write. And that's only the beginning.

"Suppose that we use the word 'element' to refer to whatever it is we are talking about — taps, fingers, booms, hands, strings, labels, points, etc.

"We say that all collections or sets of elements that can be put in 1-to-1 correspondence with each other have the same number of elements, and we assign some symbol — mark or word — to indicate that number. Sometimes we call this word or mark a 'numeral.' "

"Aha!" X-Nighyun broke in. "What you started me off with was a game that you can play with numerals and tables. And now, you're going to teach me the game of numbers. Is that it?"

"You could put it that way."

"And I'll bet when I write out a play in the game of numerals, it will also make sense if I interpret it as a play in the game of numbers and things."

"You could put it that way."

"When two games are so much alike, you must have a lot of trouble telling one from the other."

"That's right."

Had you realized that you can play both these games — and an observer would find it most difficult to tell which game you were playing if all he saw were the results you wrote down?

A.	I	II	III	IV
m	-4			
n	4	6		.5
$m + n$		6	2	.75
$m - 2n$	-12		-1	

B.	I	II	III	IV
r	19			$5\frac{2}{3}$
s	4	10		
$(r - s) + 3$	18		40	
$r + (s - 3)$	20	7	34	4

C.	I	II	III	IV
p	49	.5	$\frac{1}{2}$	
q	100		$\frac{1}{3}$	
$2p + q$				-7
$p - q$		.45		7

D.	I	II	III	IV
a	9			
k	7	10		
ak	63	5	.9	-28
$k - a$			2.7	11

Putting a Number Line to Use

Thus far, your discussions over the space phone have depended heavily on language. It would be most helpful to have something concrete to refer to — something other than hands and strings and booms — something that helps visualize ideas.

Let's try a number line.

"We shall begin today by drawing a straight line, and we'll think of what we draw as a part of a number line."

"What is a line, and what do you mean by straight?"

Harder than you thought, but not impossible. How would you explain?

(We assume that you have succeeded.)

"Mark a point on that line and label it with a 0. Mark another point on the line to the right of the first point and label it with a 1."

"Not so fast! I've got the first part — marking a point and labeling it 0. But which way is to the right and how far away shall I put the second mark?"

How can you explain "right" and "left" over the space phone? How can you describe a particular distance?

We suggest you avoid both these questions.

"Which way is to the right? Just take your pick, and remember which way you call 'to the right' and select any distance to the right you wish."

"Done."

"Put marks to the right and left of 0 so that each is the same distance from one of its neighbors as it is from the other.

"What do you mean by 'same distance'?"

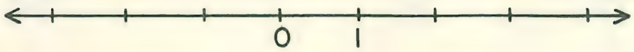
If you only had a chalkboard and television rather than a space phone!

A.	I	II	III	IV
r	4	$\frac{1}{4}$	-2	1.7
s	6		-3	
$2r + s$		$3\frac{1}{2}$		
$2(r + s)$				
$2r + 2s$				
$r - 2s$				
$(r - 2)s$				-18
$3s - r$				
$3(s - r)$				

B.	I	II	III	IV
b	6			
g	14	.3	5	
$2bg$		.48		$\frac{1}{2}$
$2b + g$			$5\frac{2}{3}$	
$2 + bg$				$2\frac{1}{4}$
$(2 + b)g$				
$2b - g$				
$2(b - g)$				
$2b - 2g$				

Here's one attempt at an answer to X-Nighyun's question: Place marks so that every chunk of line that would just include three marks on it would look exactly like every other such chunk. Would that do the job?

(We assume that you succeeded.)



"Our next job is to label all the points you marked to the right of the point labeled 0."

"Don't tell me. I've already invented labels I can use. But what about the points the other way from 0?"

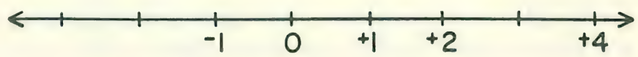
"We would say, 'to the left of zero.' We would use the same labels going in that direction."

"Then you must say 'two to the right' and 'two to the left,' etc."

"Yes; we do that by adding a mark to indicate direction. If we use a raised cross mark to indicate 'to the right' then we use a raised dash to indicate 'to the left' — and we would read these labels 'positive two' and 'negative two.' But don't ask me what these marks look like — you can invent your own."

"Done."

So, X-Nighyun must have a drawing that has the same characteristics as the following:



"Now I'm ready to assign you some homework."

"Good!"

"We have labeled only a few points on the line. Our task is to find labels for all points on the line."

"But there are an unlimited number."

"I didn't say it was easy."

A.	I	II	III	IV
a	1		-1	$\frac{1}{2}$
b	2	7		
c	3	5		
$a + b$	3		-3	
$b + c$	5		1	$\frac{3}{8}$
$a + c$	4	7		$\frac{5}{8}$
$2a + b$	4			
$b + 2c$	8			
$3a + 4c$	15			

B.	I	II	III	IV
r			2.5	
s	-1			
t			.08	
$r + s$	3	$\frac{5}{6}$		$-1\frac{1}{2}$
$s + t$		$\frac{7}{12}$		$7\frac{1}{2}$
$r - t$	-1			-9
$2(r - t)$		$\frac{1}{2}$		
$3s + 3t$			3.24	
$\frac{r + s}{2}$				

X-Nighyun Does His Homework

"I've got a system. I've got a system to name all the points on the line. It goes like this.

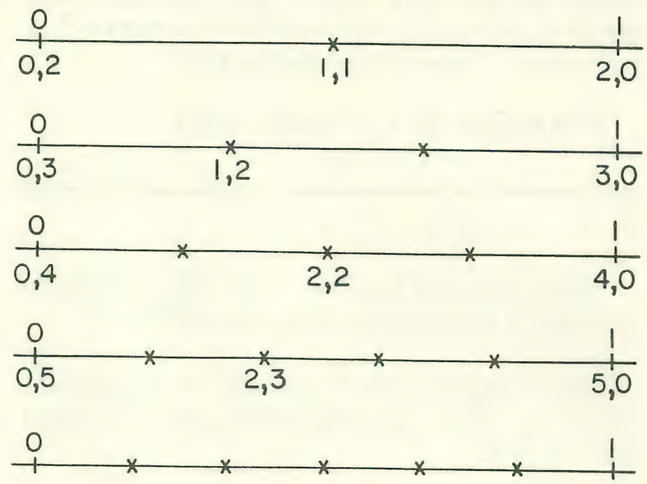
"On the piece of line from 0 to 1, you mark a point that divides the piece into two pieces of the same length. Or, you can mark two points on the piece of line from 0 to 1 to give you three pieces of the same length. Or, you can mark three points, etc.

"Each time, you can label the marked points by noting the number of pieces between 0 and the point and noting the number of parts between the point and 1.

"That will do it for a lot of points between 0 and 1."

"Slow down. Let me work out some examples."

Do you see X-Nighyun's system from the following examples? *(Please fill in the missing labels.)*



"I can divide the distance between 0 and 1 into as many parts of the same length as I like, and then count the parts on each side of my point.

A.	I	II	III	IV
x				
y				-1
z				
$x + y + 5$	11			
$y + z - 2$	1			-1
$x + z$	7	10		
$3x - y - 4$	10			
$x - 2z$	1	1		-1
$2y + 3z$			16	
$x + y + z$	8	14	8	
$2x - y + 3z$			$^-8$	

B.	I	II	III	IV
k				
m				
b				
$k + m + b$	12		-12	
$k + m - b$	-2			
$k - m + b$	8			
$k - m - b$			+6	
$-1k + m + b$		$^+\frac{1}{3}$		
$-1k - m + b$		$^-\frac{2}{3}$	-2	$-6\frac{1}{2}$
$-1k + m - b$				$-7\frac{1}{2}$
$-1k - m - b$				$-7\frac{1}{2}$

"Of course, I get several labels for each point, so I put an equal sign between them and say that one is as good as another. I also get new labels for 0 and 1."

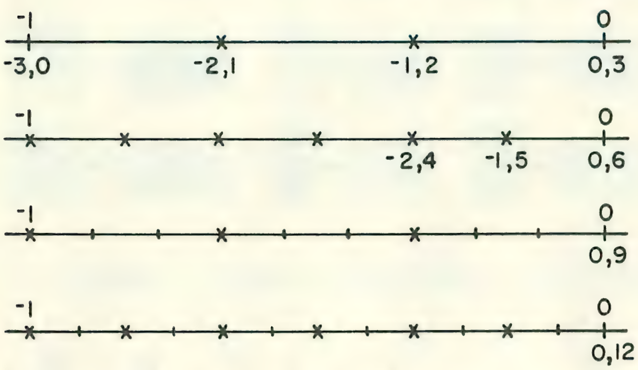
0 = 0,2 = 0,3 = _____ = _____

1 = 2,0 = 3,0 = _____ = _____

1,1 = _____ = _____

1,2 = _____ 2,1 = _____

"On the left side of 0, I use the same kind of system. I note first the number of parts between 0 and the point, and then the number of parts between the point and ¯1. I put a 'raised dash' before each label." *(Please label the cross marks.)*



-3,0 = -6,0 = _____ = ____

-1,2 = _____ = _____ = _____

-2,1 = _____ = _____ = _____

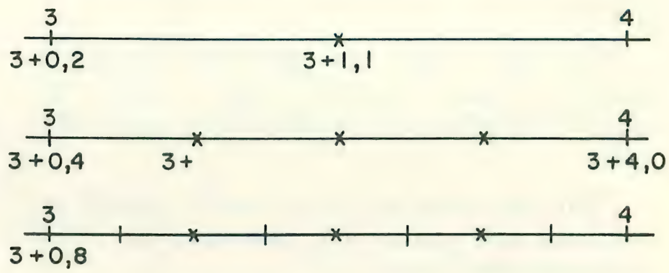
-3,3 = _____ -5,1 = _____

-1,5 = _____ 0,3 = _____ = _____

"Now, I am ready to consider points between 1 and 2, 2 and 3, 3 and 4, etc.

"For example, consider points that divide the piece between 3 and 4 into pieces of the same length. To label any of these points, I first write a 3, then a plus sign, and then a pair of numbers which, as before, tells the numbers of parts on each side of the point."

(Please label the cross marks.)



3 + 0,2 = _______ = _______

3 + 1,1 = _______ = _______

3 + 1,3 = _______ = _______

3 + 3,1 = _______ = _______

What do you think of X-Nighyun's system?

A.	I	II	III	IV
a				
b				
c				
$2a+b+c$	48	16	25	96
$a+2b+c$	40	9	0	80
$a-b+2c$	30	5	-25	60

B.	I	II	III	IV
p				
q				$\frac{3}{8}$
r				
$2p+q+r$	22	36	4	$2\frac{3}{8}$
$p+2q-r$	-1	18	-25	$\frac{1}{4}$
$p-q+2r$	23	18	29	$2\frac{1}{8}$

You Make a Request

"And now let me tell you about using these labels in your addition game and . . ."

"Wait a minute, friend. I have decided your set of symbols is just as good as mine, but I have a request. Since I'm very familiar with my symbols and you are just getting started, please use the same symbol system I do. It will be a little hard on you at first. However, the systems are basically the same."

"Fair enough."

"I start out by dividing the part of the line between 0 and 1 into a certain number of parts of the same length and label each division point. But, I record what I see in a little different way.

"Beneath a short horizontal line I note the number of parts between 0 and 1. Above the line, I note the parts from 0 to the particular point."

"All right with me. I can easily change my symbols into yours. For example, my '2, 3' becomes your '5 under 2'."

"Or, 2 over 5 as we usually describe it . . . or, 'two-fifths' as we usually read it."

Here are some examples that show the problem one faces in converting from one of these systems to another:

More examples, but without sketches:

4,3 → —	0,6 → —	2 + 1,3 →
, → $\frac{2}{11}$	5,0 →	5 + 7,1 →
,13 → $\frac{5}{\ }$	, → $\frac{7}{14}$	+ , → $4\frac{4}{8}$

A.	I	II	III	IV
p				
q				
r				
s				
$p + q$	6	7		2
$r + q$	4	5	22	-2
$r + s$	11	3	39	-1
$p + s$	13	5	31	
$p + r$	8	6	10	2

B.	I	II	III	IV
g		2		
h				
m				
k				$\frac{1}{8}$
$g - h$	3	-6	5	$\frac{1}{2}$
$m - h$	-2	-7	12	$\frac{-1}{4}$
$g + m$	15	3	53	$1\frac{1}{4}$
$h - k$	6	-3	9	$\frac{3}{8}$
$g + h + m + k$	23	22	80	$1\frac{7}{8}$

X-Nighyun Is Eager

"Done! I agree to use your symbols and I'll just file my equally good system.

"I wish to tell you about using these new symbols or labels in your addition game. First, we'll make a chart. Across the top we shall begin listing all possible labels."

"Wait a minute. You couldn't ever list all the labels I could make up for different points between 0 and 1."

"No, but I can make up a list which, if extended by my rules, would give you any label you can think of."

"How?"

"Instead of explaining my rules, it would be easier to start the list and let you see the pattern I'm following. It starts '0 over 1, 0 over 2, 1 over 1, 0 over 3, etc.'

$\frac{0}{1}$	$\frac{0}{2}$	$\frac{1}{1}$	$\frac{0}{3}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{0}{4}$	$\frac{1}{\ }$
$\frac{2}{\ }$	$\frac{3}{\ }$	$\frac{0}{5}$	$\frac{\ }{4}$	$\frac{\ }{3}$	$\frac{\ }{2}$	$\frac{\ }{1}$	$\frac{0}{6}$
$\frac{\ }{\ }$	$\frac{\ }{\ }$	$\frac{\ }{\ }$	$\frac{\ }{\ }$	$\frac{\ }{\ }$	$\frac{0}{7}$	$\frac{\ }{\ }$	$\frac{\ }{\ }$

"If I extended this list, I would include any label you could think up."

"Next, repeat the list down the side of the chart.

"Now, addition of labels means you look up one label in the top list, the other in the side list, and then find the entry in the block that's in that column and that row. Done!"

"But there aren't any entries in my chart."

As if he hadn't heard, X-Nighyun went on, "Then you make a similar chart that has different entries and call it the multiplication chart."

A.	I	II	III	IV
a				
b				
c				
d				
$a+b+c+d$	14	11	$2\frac{1}{2}$	-14
$a+b+c-d$	2	1	1	-2
$a+2b-c+d$	7	9	$2\frac{1}{2}$	-7
$a-b+c+3d$	24	25	3	-24

B.	I	II	III	IV
w				
x				
y				
z				
$(w+x)-(y+z)$	12	11	1	0
$w-(x+y+z)$	4	1	-15	-6
$(w-x)+(y-z)$	6	-3	-15	0
$w-(x-y-z)$	18	5	35	6

"But, again, I have no entries — only labels for the columns and the rows."

"Oh, make any entries you wish provided you follow a few simple rules:

(a) Every entry in both charts must be one of your labels;

(b) both charts must, when used, demonstrate the commutative and associative principles;

(c) each must have an identity element — and not the same one;

(d) and multiplication must distribute over addition.

"Of course, you are familiar with these rules because you explained them to me.

"Now as long as you don't violate these rules, make up any entries you like."

Then, as an afterthought, he added:

"And if an entry in the multiplication chart is a label for zero then the label at the top of that column or at the side of that row must also be a label for zero.

"Now it's your turn to do some homework!"

How should you begin?

+	$\frac{0}{1}$	$\frac{0}{2}$	$\frac{1}{1}$	$\frac{0}{3}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{0}{4}$
$\frac{0}{1}$	—	—	—	—	—	—	—
$\frac{0}{2}$	—	—	—	—	—	—	—
$\frac{1}{1}$	—	—	—	—	—	—	—
$\frac{0}{3}$	—	—	—	—	—	—	—
$\frac{1}{2}$	—	—	—	—	—	—	—

A.	I	II	III	IV
x		$\frac{1}{2}$		10
y			.7	1.5
z		$\frac{2}{3}$		
$x - y$	2		-3.3	
$y - z$	-6		-.35	
$x + z$	10			
$4y - 7$		13		
$xy + z$				15
$x^2 + 2y + z$				

B.	I	II	III	IV
k				
m		12		
q			2	
$2k - m$	10		9	-5
$3q - k$	18			$1\frac{1}{2}$
$q + \frac{m}{2}$	9	9		$3\frac{2}{3}$
kmq		36		2
$\frac{kq}{m}$				
$\frac{k+m}{q}$			0	

Just to show X-Nighyun how easily that can be done, you decide to add numerators and add denominators to find entries for the addition chart, and multiply numerators and multiply denominators for the entries in the multiplication chart. Since it pains you to write that $\frac{1}{2} + \frac{1}{3}$ equals $\frac{2}{5}$ — rather than $\frac{5}{6}$ — you decide to put a circle around any addition sign between these labels.

You don't even need to make charts because you can imagine them as you go. So, when you call him back you are ready to demonstrate your solution to his problem:

(a) Every pair of labels leads to an entry that is a label:

$$\frac{1}{3} \oplus \frac{2}{5} = \frac{3}{8} \qquad \frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$$

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

(b) The commutative and associative principles hold.

$$\frac{2}{3} \oplus \frac{1}{7} = \frac{1}{7} \oplus \frac{2}{3}$$

$$\frac{2}{3} \times \frac{1}{7} = \frac{1}{7} \times \frac{2}{3}$$

$$\left(\frac{1}{2} \oplus \frac{3}{4}\right) \oplus \frac{1}{3} = \frac{1}{2} \oplus \left(\frac{3}{4} \oplus \frac{1}{3}\right)$$

$$\left(\frac{1}{2} \times \frac{3}{4}\right) \times \frac{1}{3} = \frac{1}{2} \times \left(\frac{3}{4} \times \frac{1}{3}\right)$$

"That was an easy assignment," you tell X-Nighyun.

"Not as easy as you think, my friend. Your pattern for addition holds up for quite a ways . . . but show me your identity element. How about my rule (c)? Complete the following using your table:

$$\frac{1}{2} \oplus \_\_\_ = \frac{1}{2} \qquad \frac{2}{5} \oplus \_\_\_ = \frac{2}{5}.$$

You answer quickly that

$$\frac{1}{2} \oplus \frac{0}{0} = \frac{1}{2} \qquad \frac{2}{5} \oplus \frac{0}{0} = \frac{2}{5}.$$

"There is no such label as $\frac{0}{0}$, and it would not be an appropriate label on the number line you described to me.

A.	I	II	III	IV
$p + q$	21	10	60	0
$p - q$	9	-4	-8	.4
p				
q				

B.	I	II	III	IV
$x + 2y$	14	27	36	42
$x - y$	5	-3	0	9
x				
y				

C.	I	II	III	IV
$a + 2b$	13	27	66	$1\frac{1}{6}$
$2a + 2b$	20	36	82	$1\frac{2}{3}$
a				
b				

D.	I	II	III	IV
$2r + t$	23	29	4	1.3
$r + 2t$	16	37	-4	1.1
r				
t				

"Try again and, I might add, your system would fail again in that it would not follow rule (d) — which you can check for yourself.

"Back to the asteroid pile, as we say on Mars."

"I'll talk to you tomorrow."

Another Attempt

We learned our lesson.

Rather than start all over, let's keep what we already have, trying variations.

Let's see what happens if we make up a new rule for $\oplus$ — you add numerators and multiply denominators. This would lead us to write:

$$\frac{1}{2} \oplus \frac{3}{5} = \frac{1+3}{2 \times 5} = \frac{4}{10}$$

$$\frac{3}{7} \oplus \frac{2}{3} = \text{—} \qquad \frac{7}{9} \oplus \frac{2}{3} = \text{—}$$

In general: $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b \times d}$

Does this pass X-Nighyun's rules (a) and (b)? ______

Is there a label that we can use as an identity element so we can complete the following?

$$\frac{1}{2} \oplus \_\_\_ = \frac{1}{2} \qquad \frac{2}{5} \oplus \_\_\_ = \frac{2}{5}$$

Finally, does the distributive principle hold? Consider the following pattern:

$$\left(\frac{a}{b} \oplus \frac{c}{d}\right) \times \frac{m}{n} = \frac{a \times m}{b \times n} \oplus \frac{c \times m}{d \times n}$$

Let's try one example: substitute 1 for a, 2 for b, 3 for c, 4 for d, 1 for m, and 3 for n.

$$\left(\text{—} \oplus \text{—}\right) \times \text{—} = \frac{\times}{\times} \oplus \frac{\times}{\times}$$

$$\left(\frac{\quad}{8}\right) \times \frac{1}{3} = \frac{1}{\quad} \oplus \frac{\quad}{12}$$

$$\frac{\quad}{24} = \frac{4}{\quad}$$ (a counter-example)

A most unhappy result!

A.	I	II	III	IV
$a - b + 3$	7	12	-13	$13\frac{3}{4}$
$a + b - 3$	3	44	13	$11\frac{1}{4}$
a				
b				

B.	I	II	III	IV
$2p + q$	23	97	30	$2\frac{5}{8}$
$p - q + 7$	14	18	-2	7
p				
q				

C.	I	II	III	IV
$xy - 1$	15	80	$1\frac{1}{2}$	.05
$x - y$	6	0	$4\frac{1}{2}$	.8
x				
y				

D.	I	II	III	IV
$2ak$	182	25	4.2	$\frac{1}{6}$
$2a + k$	27	26	6.7	$\frac{11}{12}$
a				
k				

Try as you will to work out some other way to follow X-Nighyun's rules, you will almost certainly be forced to a rule you already know:

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a \times d + b \times c}{b \times d}$$

Does our rule for multiplication meet all of X-Nighyun's requirements? ________

Do our rules for multiplication meet all of X-Nighyun's requirements? ________

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

So, this time when X-Nighyun calls, we are ready for him. We explain our rules for making entries in the two charts — the one marked "addition" and the other marked "multiplication."

"You may be surprised, but I have exactly the same rules you have. At least, they would lead to exactly the same entries in the two charts. So we are together.

"But I'm troubled that we have limited our discussion to the labels on the 'right' side of our number line — the ones you indicate with a 'raised cross' when you wish to distinguish them from the others on the left, the ones you indicate with a 'raised dash'."

"You mean labels for what we refer to as positive and negative numbers?"

"Yes. But I'm not sure I know any reason to keep talking about labels and numbers. You can be sure that I shall know when you are talking about labels and when you are talking about numbers. If you are worried in any particular case then let me know — and if I have questions, I'll ask them!"

A.	I	II	III	IV
$2(a+b)$	22	32	150	0
$a+2b$	14	31	125	-4
a				
b				

B.	I	II	III	IV
$(x-y)3$	36	-18	21	$\frac{3}{4}$
$x+y$	18	8	61	$\frac{3}{4}$
x				
y				

C.	I	II	III	IV
$4m-n$	3	18	18	1
$n-m$	3	-3	3	$\frac{1}{2}$
m				
n				

D.	I	II	III	IV
$3r-2t+7$	10	26	14	37
$r+2t-3$	14	14	18	7
r				
t				

"Quite a lecture! But you're probably right. Before continuing, here is a little test of your ability to use our Earth arithmetic."

Here are your questions for X-Nighyun.

1. "Rockton and Newton are 945 miles apart by railroad. A slow freight train — 'S' for short — leaves Rockton and travels 45 miles per hour. How long will it take for S to reach Newton?"

Answer: ____________

2. "A fast freight train — 'F' for short — leaves from Newton and travels at the rate of 60 miles per hour. How long will it take to reach Rockton?"

Answer: ____________

3. "If S left Rockton at 1 P.M., what time did it arrive at Newton?"

"I suppose it arrived at 22 P.M., whatever 'P.M.' means."

Always more explaining to do. Could you make X-Nighyun understand our system of talking about time?

4. "All right, I'll change my question. If S left from Rockton and F from Newton at the same time, how much earlier would F reach Rockton than S reached Newton?"

Answer: ____________

5. "How far would S be from Newton when F reached Rockton?"

Answer: ____________

"S left from Rockton and F left from Newton at 1 P.M. on the same day. Now I have several questions to ask:

6. "How far apart would they be at 2 P.M.?"

Answer: ____________

7. "How far apart at 5 P.M.?"

Answer: ____________

8. "How far apart at 7 P.M.?"

Answer: ____________

9. "How far apart at 1 A.M.?"

Answer: ____________

10. "At what time did they meet?"

Answer: ____________

11. "How far apart at 4 A.M.?"

Answer: ____________

12. "How much further has F to go?"

Answer: ____________

13. "How much further has S to go?"

Answer: ____________

14. "When S reached Junction City, 50% of its fuel had been used up. When it got to Newton, there was still 20% left. About how far is Junction City from Rockton?"

Answer: ____________

15. "Of course, you could only make a reasonable estimate. Actually, Junction City is 245 miles further from Rockton than from Newton. Now can you tell me how far it is from Junction City to Newton."

Answer: ____________

16. "Now, here is a brainbuster! S left Rockton at 1 P.M. and traveled at the rate of 45 miles per hour. S met F at Junction City; and F had traveled at the rate of 60 miles per hour. At what time did F leave Newton?"

Answer: ____________

"That sure was a brainbuster. I can work more easily with glooms per glud. How about you?"

You shouldn't be surprised, one day, if X-Nighyun started asking some strange questions. Like these:

"Now that I understand something about your arithmetic, I'm going to find out something about how clever you are.

"If I made a trip to Astra A from here near your planet's moon, and then returned, it would be 750,000,000 glooms. How far is it from here to Astra A?"

"But how can I tell when I don't know what a gloom is?"

"If you told me that a round trip between New York and Chicago is 1680 miles, I could tell you that those cities are 840 miles apart . . . and I have no idea about the length of your mile."

"Oh, you want to know how many glooms it is from you to Astra A?"

"Of course."

"The distance must be about

______________________."

"I once made a trip starting from Astra A and going to Astra C, stopping at Astra B on the way. The total trip was 204,000 glooms. It is twice as far from Astra A to Astra B as it is from Astra B to Astra C. How far is it from Astra A to Astra B?"

Answer: ______________________

"I would have saved 17,500 glooms by going straight from Astra A to Astra C. What is the shortest distance from Astra A to Astra C?"

Answer: ______________________

"How far is it from here to Astra B if I stop at Astra A on the way?"

Answer: ______________________

"If my speed is 50,000 glooms per glud, how many gluds will it take for me to get to Astra A from here?"

Answer: ______________________

"If I returned immediately at half that speed, how long would I have been gone?"

Answer: ______________________

"On your moon, I weigh 18.5 chorns when I have my space suit on. Now, 40% of that weight is my suit. How much do I weigh?"

Answer: ______________________

"When I am in my suit and carrying another suit, how much would the total weight be?"

Answer: ______________________

"A shipment of these space suits weighed 555 chorns. How many space suits were in the shipment?"

Answer: ______________________

"At 20,000 glooms per glud, how long would it take me to go from Astra C to Astra B?"

Answer: ______________________

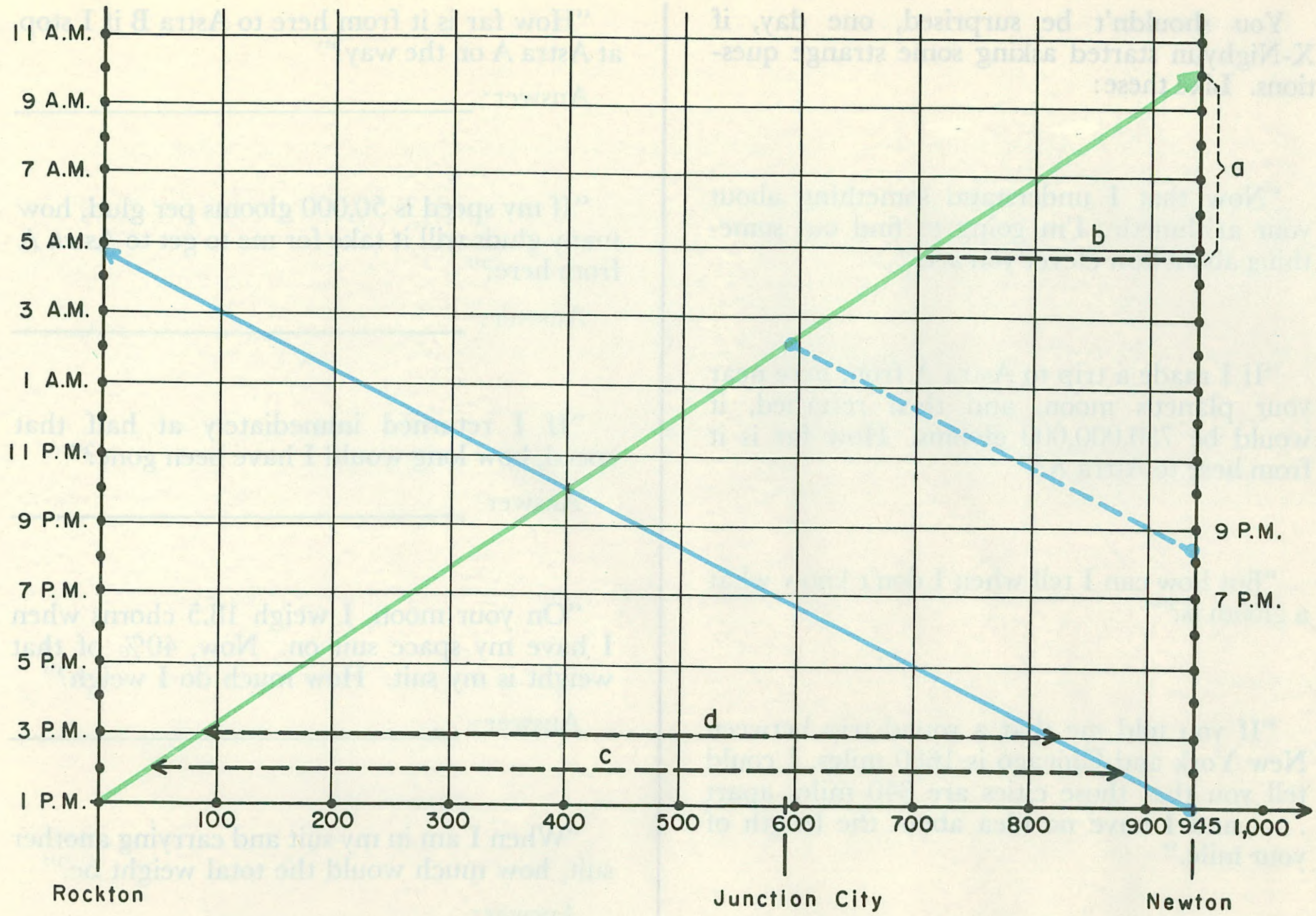
"At the same speed, how long from Astra B to Astra A?"

Answer: ______________________

"At the same speed, how long from Astra A to my present position near your moon?"

Answer: ______________________

"Now it's your turn!"



By interplanetary mail, X-Nighyun sent a note along with the sketch above.

While you were asking all those questions about the two freight trains, I used the arithmetic you taught me to find the answers. But I also made a diagram to help me out.

The diagram contains much of the information you gave and much of the information you made me figure out.

Another important use for the diagram is that it keeps me from making any unreasonable answers. I checked the answer to every problem — except one — by estimating the answer with the diagram above. The exception was that one about fuel.

I've marked parts of the sketch that I used to help with your third, fourth, and fifth questions.

I needed to answer the first two questions before I could draw the sketch.

I'm going to let you decide how I used this diagram to find out whether any answers were unreasonable.

All I did to test my answer to your brainbuster was to draw the broken blue line.

Thanks for your explanations about arithmetic. I know there is more, but I wish to spend some time putting together the pieces you gave me. The patterns help me see what's going on.

Sincerely,

X-Nighyun

Can you discover ways in which X-Nighyun used his sketch?

1 2 3 4 5 6 7 jp 73 72 71 70 69 68 67